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ASCENDING AND ASYMPTOTIC SERIES FOR SQUARES, PRODUCTS AND CROSS-PRODUCTS OF MODIFIED BESSEL FUNCTIONS

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US ARMY ARMAMENT RESEARCH AND DEVELOPMENT COMMAND
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Asymptotic series for squares and products of modified Bessel functions were programmed as well as the ascending and asymptotic series for the cross products. These programs reduced round off error and led to precise calculation of stresses in thin-walled cylinder according to the equations of elasticity.

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I. INTRODUCTION

In the Fourier integrals used to calculate stresses in a hollow cylinder, the variable of integration ranges from zero to infinity along the real axis; consequently the Bessel functions in the integrand must be computed for very large and very small values of the argument. Calculation of stresses in a thin-walled cylinder by residue theory requires Bessel functions of complex argument and very large modulus. Use of the Bessel function previously described[1] to calculate the integrand for these extreme ranges leads to exponential overrun and cancellation of significant digits. Axial stresses due to discontinuous internal shear loading are calculated from a Fourier integral with an apparent singularity at the lower limit of integration. Evaluation of the integral by quadratures leads to the numerical difference of nearly equal numbers for points near the discontinuity of loading.

Exponential overrun is especially critical in calculating stresses for thin-walled cylinders according to the theory of elasticity. Asymptotic series have been used to calculate stresses for cylinders with wall ratios in the range $1.01 \leq B \leq 1.25$. The lower part of this range is not accessible when conventional tables or subroutines are used to calculate the integrand of the Fourier integral.

This report describes three algorithms which have been used to overcome the numerical difficulties described above. The corresponding computer programs are listed in the thick-walled cylinder code[2] and will not be repeated.

II. SOURCES OF NUMERICAL DIFFICULTIES

The characteristic function for a hollow cylinder with stress-free cylindrical surfaces illustrates the numerical difficulties in evaluating determinants arising in thick-walled cylinder theory.

$$\Delta(s) = \begin{vmatrix} I_0(p) & K_0(p) & \alpha_1 I_1(p) & \alpha_1 K_1(p) \\ I_1(p) & -K_1(p) & pI_0(p) & -pK_0(p) \\ I_0(q) & K_0(q) & \beta_1 I_1(q) & \beta_1 K_1(q) \\ I_1(q) & -K_1(q) & qI_0(q) & -qK_0(q) \end{vmatrix} , \quad (1)$$

where

[1] *K.L. Zimmerman, A.S. Elder, A.K. Depue, "User's Manual for the BRL Subroutine to Calculate Bessel Functions of Integral Order and Complex Argument", Ballistic Research Laboratory Report No. ARBRLTR-02068, May 1978.* (AD A056369)

[2] *A.S. Elder and K.L. Zimmerman, "Stresses in a Gun Tube Produced by Internal Pressure and Shear", BRL Memorandum Report No. 2495, June 1975. AD #A012765.*

$$
\begin{aligned}
p &= sa \ , \\
q &= sb \ , \\
\alpha_1 &= p + (2-2\nu)/p \ , \\
\beta_1 &= q + (2-2\nu)/q \ ,
\end{aligned}
$$

and

$$
\begin{aligned}
\nu &= \text{Poisson's ratio} \ , \\
a &= \text{inner radius of cylinder} \ , \\
b &= \text{outer radius of cylinder.}
\end{aligned}
$$

If we evaluate $\Delta(s)$ by approximating the Bessel functions with Hankel asymptotic series, we find products of the type

$$e^{sa}\ e^{-sb} \text{ and } e^{-sa}\ e^{sb}$$

occurring, leading to exponential overrun when $|sa|$ and $|sb|$ are very large. We also find, after considerable algebra, that the first and second terms of the Hankel asymptotic series are lost by subtraction, leading to a loss of significant figures.

The characteristic function $\Delta(s)$ has a pole at the origin, but is analytic elsewhere in the finite part of the complex s plane. However, the Bessel functions of the second kind in Eq. (1) have logarithmic singularities at the origin. The logarithmic singularities will cause a loss of significant figures when $|s|$ is small. The algorithm for calculating cross products of the modified Bessel functions for small $|s|$ eliminates this difficulty.

III. ASCENDING SERIES FOR CROSS PRODUCTS OF MODIFIED BESSEL FUNCTIONS

Cross products for the modified Bessel functions are defined by the following equations[3]

$$
\begin{aligned}
L_{0,0}(s) &= I_0(sa)K_0(sb)-K_0(sa)I_0(sb) \\
L_{0,1}(s) &= I_0(sa)K_1(sb)+K_0(sa)I_1(sb) \\
L_{1,0}(s) &= I_1(sa)K_0(sb)+K_1(sa)I_0(sb) \\
L_{1,1}(s) &= I_1(sa)K_1(sb)-K_1(sa)I_1(sb) \ ,
\end{aligned}
\qquad (2)
$$

[3] *S. Timoshenko and J.N. Goodier, Theory of Elasticity, Second Edition, McGraw-Hill Book Company, New York, 1951.*

where s is the variable of integration in the Fourier integrals. We now show that the $L_{1,0}(s)$ functions are analytic in s except for a simple pole at the origin. We have[4]

$$I_0(sa) = 1 + \frac{(sa)^2}{2^2(1!)^2} + \frac{(sa)^4}{2^4(2!)^2} + \frac{(sa)^6}{2^6(3!)^2} + \ldots$$

$$I_0(sb) = 1 + \frac{(sb)^2}{2^2(1!)^2} + \frac{(sb)^4}{2^4(2!)} + \frac{(sb)^6}{2^6(3!)^2} + \ldots$$

$$I_1(sa) = \frac{(sa)}{2} + \frac{(sa)^3}{2^3 1!2!} + \frac{(sa)^5}{2^5 2!3!} + \ldots \tag{3}$$

$$I_1(sb) = \frac{(sb)}{2} + \frac{(sb)^3}{2^3 1!2!} + \frac{(sb)^5}{2^5 2!3!} + \ldots ,$$

for modified Bessel functions of the first kind. In the series for modified Bessel functions of the second kind, we separate the logarithmic terms to facilitate the algebraic manipulation. For the outside radius we have[5]

$$K_0(sb) = G_0(sb) - M_0(sb) - S_0(b,s) , \tag{4}$$

where

$$G_0(sb) = \sum_{r=1}^{\infty} \frac{1}{(r!)^2} \left(\frac{sb}{2}\right)^{2r} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{r}\right) , \tag{5}$$

$$M_0(sb) = I_0(sb) \log_e \left(\frac{b}{2}\right) , \tag{6}$$

$$S_0(b,s) = I_0(sb) \left[\gamma + \log_e s\right] , \tag{7}$$

[4] *V.K. Prokopov, "Equilibrium of an Elastic Axisymmetrically Loaded Thick-Walled Cylinder", Priladnaya matematika i mehanika, Vol. XIII, 1949, pages 135-144. Institute of Mechanics of the Academy of Sciences, USSR. FTIO Translation No. J-2589, Aberdeen Proving Ground, Maryland, Translation dated 22 August 1967.*

[5] *British Association for the Advancement of Science, Bessel Functions, Part I, Function of Orders Zero and Units, Mathematical Tables, Volume VI, University Press, Cambridge, 1937.*

where γ is Euler's constant.

Similarily we have

$$K_1(sb) = -G_1(sb) + M_1(sb) + S_1(b,s) \ , \qquad (8)$$

where

$$G_1(sb) = \sum_{r=1}^{\infty} \frac{1}{(r-1)!r!} \left(\frac{sb}{2}\right)^{2r-1} \left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{r}-\frac{1}{2r}\right), \qquad (9)$$

$$M_1(sb) = I_1(sb)\ \log_e \left(\frac{b}{2}\right) + \frac{1}{sb} \ , \qquad (10)$$

$$S_1(b,s) = I_1(sb)\ [\gamma + \log_e s] \ . \qquad (11)$$

Similar formulas are obtained for the inside radius.

$$K_0(sa) = G_0(sa) - M_0(sa) - S_0(a,s) \ , \qquad (12)$$

$$G_0(sa) = \sum_{r=1}^{\infty} \frac{1}{(r!)^2} \left(\frac{sa}{2}\right)^{2r} \left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{r}\right) \ , \qquad (13)$$

$$M_0(sa) = I_0(sa)\ \log_e \left(\frac{a}{2}\right) \ , \qquad (14)$$

$$S_0(a,s) = I_0(sa)\ [\gamma + \log_e s] \ , \qquad (15)$$

and

$$K_1(sa) = -G_1(sa) + M_1(sa) + S_1(a,s) \ , \qquad (16)$$

where

$$G_1(s,a) = \sum_{r=1}^{\infty} \frac{1}{(r-1)!r!} \left(\frac{sa}{2}\right)^{2r-1} \left(1 + \frac{1}{2} + \frac{1}{3} \cdots \frac{1}{r} - \frac{1}{2r}\right), \tag{17}$$

$$M_1(sa) = I_1(sa) \log_e \left(\frac{a}{2}\right) + \frac{1}{sa} , \tag{18}$$

$$S_1(a,s) = I_1(sa) \; [\gamma + \log_e s] \; . \tag{19}$$

We now eliminate Bessel functions from the formulas for the cross products. We obtain

$$L_{0,0}(s) = I_0(sa)[G_0(sb)-M_0(sb)]-I_0(sb)[G_0(sa)-M_0(sa)] \tag{20}$$

$$L_{0,1}(s) = I_0(sa)[-G_1(sb)+M_1(sb)]+I_1(sb)[G_0(sa)-M_0(sa)] \tag{21}$$

$$L_{1,0}(s) = I_1(sa)[G_0(sb)-M_0(sb)]+I_0(sb)[-G_1(sb)+M_1(sb)] \tag{22}$$

$$L_{1,1}(s) = I_1(sa)[-G_1(sb)+M_1(sb)]-I_1(sb)[-G_1(sa)+M_1(sa)] \; . \tag{23}$$

The logarithmic singularity in s is eliminated. The only remaining singularity is a simple pole at the origin due to the functions $M_1(sa)$ and $M_1(sb)$.

The first few terms of the ascending series for the $L_{i,j}(s)$ functions can be obtained by multiplication of the appropriate series. We find

$$L_{0,0}(s) = -\log_e(b/a) + [-(a^2+b^2)\log_e(b/a)+(b^2-a^2)]s^2/4 + \ldots \tag{24}$$

$$L_{0,1}(s) = 1/(sb) + [2b \log_e (b/a) - (b^2-a^2)/b]s/4 + \ldots \tag{25}$$

$$L_{1,0}(s) = 1/(sa) + [-2a \log_e (b/a) + (b^2-a^2)/a]s/4 + \ldots \tag{26}$$

$$L_{1,1}(s) = -(b^2-a^2)/(2ab)+[ab\ \log_e(b/a)-(b^4-a^4)/(4ab)]s^2/4 + \ldots \tag{27}$$

Calculation of literal coefficients for higher powers of s requires excessive algebraic manipulation. Numerical values are readily calculated from recursion formulas, using the first terms in the above expansions as starting values.

The recursion formulas for the coefficients are derived from a set of first-order, linear differential equations satisfied by the cross products.

$$L'_{0,0}(s) = aL_{1,0}(s) - bL_{0,1}(s) \tag{28}$$

$$L'_{0,1}(s) = aL_{1,1}(s) - bL_{0,0}(s) - L_{0,1}(s)/s \tag{29}$$

$$L'_{1,0}(s) = aL_{0,0}(s) - bL_{1,1}(s) - L_{1,0}(s)/s \tag{30}$$

$$L'_{1,1}(s) = aL_{0,1}(s) - bL_{1,0}(s) - 2L_{1,1}(s)/s \tag{31}$$

We assume

$$L_{0,0}(s) = \sum_{n=0}^{\infty} t_n\ s^n \tag{32}$$

$$L_{0,1}(s) = \sum_{n=-1}^{\infty} u_n\ s^n \tag{33}$$

$$L_{1,0}(s) = \sum_{n=-1}^{\infty} v_n\ s^n \tag{34}$$

$$L_{1,1}(s) = \sum_{n=0}^{\infty} w_n\ s^n \tag{35}$$

We substitute these series into the preceding differential equations, and the coefficients of each power of s are set equal to zero. The following recursion formulas are obtained:

$$(n+2)w_n - a\,u_{n-1} + b\,v_{n-1} = 0 \tag{36}$$

$$(n+2)v_{n+1} - a\,t_n + b\,w_n = 0 \tag{37}$$

$$(n+2)u_{n+1} - a\,w_n + b\,t_n = 0 \tag{38}$$

$$n\,t_n - a\,v_{n-1} + b\,u_{n-1} = 0 \tag{39}$$

The indices are shifted for convenience in computation.

$$- a\,u_n + b\,v_n + (n+3)w_{n+1} = 0 \tag{40}$$

$$- a\,t_n + (n+2)v_{n+1} + b\,w_n = 0 \tag{41}$$

$$b\,t_n + (n+2)u_{n+1} - a\,w_n = 0 \tag{42}$$

$$(n+1)t_{n+1} + b\,u_n - a\,u_n = 0 \tag{43}$$

We find that $L_{0,0}(s)$ and $L_{1,1}(s)$ are even functions of s, whereas $L_{0,1}(s)$ and $L_{1,0}(s)$ are odd functions. The initial values for the recursion formulas are obtained from Eqs. (24 - 27)

$$t_0 = - \log_e(b/a) \tag{44}$$

$$u_{-1} = 1/b \tag{45}$$

$$v_{-1} = 1/a \tag{46}$$

$$w_0 = - (b^2-a^2)/(2ab) \tag{47}$$

and

$$t_1 = 0 \tag{48}$$

$$u_0 = 0 \tag{49}$$

$$v_0 = 0 \tag{50}$$

$$w_1 = 0 \tag{51}$$

The coefficients are listed in Tables 1, 2, and 3 for several values of b and a=1. The series converge exponentially for all values of s. The convergence is very rapid when the wall ratio b/a is close to unity, facilitating calculations for thin-walled cylinders. Eqs. 24 thru 27 can be expressed as

$$L_{0,0}(s) = -\log_e(b/a) + R_{0,0}(s)$$

$$L_{0,1}(s) = 1/(sb) + R_{0,1}(s)$$

$$L_{1,0}(s) = 1/(sa) + R_{1,0}(s)$$

$$L_{1,1}(s) = -(b^2-a^2)/2ab + R_{1,1}(s)$$

The graphs of the R function for b = 1.1, 1.2, 1.25, 1.3, 1.4, and 1.5 are shown in Figures 1 thru 12.

IV. ASYMPTOTIC EXPANSIONS OF CROSS PRODUCTS FOR MODIFIED BESSEL FUNCTIONS

We obtain asymptotic expansions of the cross products by expressing the Bessel functions of Eq. (2) in terms of Hankel asymptotic expansions. We assume

$$\mathrm{Im}\,(s) > 0 \tag{52}$$

to be consistent with formulas for calculating stresses by the theory of residues. Then[6]

---

[6] *G.N. Watson, A Treatise on the Theory of Bessel Functions, The MacMillan Co. New York, 1948. See Eq. (2) Page 203, and the discussion of Stokes phenomena which follows.*

TABLE 1. COEFFICIENTS FOR THE ASCENDING SERIES FOR THE CROSS PRODUCTS OF MODIFIED BESSEL FUNCTIONS FOR WALL RATIOS B = 1.1 AND B = 1.2

A= 1.0 B= 1.1

N	T(N)	U(N)	V(N)	W(N)
1	-.1588743419E-03	.4693326165E-02	.4844910098E-02	-.1590187356E-03
2	-.7944232453E-07	.3935760112E-05	.4011566826E-05	-.7949389928E-07
3	-.1891552196E-10	.1315442950E-08	.1333494113E-08	-.1892507182E-10
4	-.2627274003E-14	.2352502925E-12	.2377571298E-12	-.2628255020E-14
5	-.2411450200E-18	.2617463830E-16	.2638065193E-16	-.2370065686E-18
6	-.7997252256E-22	.2354412782E-20	.1630183790E-20	.4008647240E-22
7	-.1351681195E-23	.9146874801E-23	-.8861974443E-23	.1180940418E-23
8	-.2181751703E-25	.1667368583E-24	-.1656697284E-24	.1938741997E-25
9	-.2804599894E-27	.2410371595E-26	-.2396871055E-26	.2523464878E-27
10	-.2937447124E-29	.2804262381E-28	-.2790205630E-28	.2669767534E-29

A= 1.0 B= 1.2

N	T(N)	U(N)	V(N)	W(N)
1	-.1216149644E-02	.1772626741E-01	.1883922160E-01	-.1220199628E-02
2	-.2432876486E-05	.5979498617E-04	.6202247747E-04	-.2438664464E-05
3	-.2317330852E-08	.8013121975E-07	.8225347858E-07	-.2321619319E-08
4	-.1287514266E-11	.5739721291E-10	.5857654137E-10	-.1289463673E-11
5	-.4682199585E-15	.2555534471E-13	.2598421407E-13	-.4688093475E-15
6	-.1202379501E-18	.7754550228E-17	.7862604873E-17	-.1200411156E-18
7	-.2694857518E-22	.1731744607E-20	.1700813475E-20	-.1932697275E-22
8	-.7566323765E-25	.8132073416E-24	-.2347629926E-24	.6082905181E-25
9	-.1020395137E-26	.8423607611E-26	-.8258783324E-26	.9167073800E-27
10	-.1172465461E-28	.1070590772E-27	-.1060221996E-27	.1064935076E-28

TABLE 2. COEFFICIENTS FOR THE ASCENDING SERIES FOR THE CROSS PRODUCTS OF MODIFIED BESSEL FUNCTIONS FOR WALL RATIOS B = 1.25 AND B = 1.3

A= 1.0 B= 1.25

N	T(N)	U(N)	V(N)	W(N)
1	-.2326337561E-02	.2696471957E-01	.2905322434E-01	-.2337952714E-02
2	-.7272388339E-05	.1424923091E-03	.1490258331E-03	-.7298330365E-05
3	-.1082414467E-07	.2986925099E-06	.3084207694E-06	-.1085418148E-07
4	-.9397140967E-11	.3344999199E-09	.3429477722E-09	-.9418479528E-11
5	-.5339749079E-14	.2327946681E-12	.2375958443E-12	-.5350011442E-14
6	-.2139433397E-17	.1103895755E-15	.1123137687E-15	-.2143045377E-17
7	-.6361461035E-21	.3794616920E-19	.3852666606E-19	-.6382602103E-21
8	-.1346635327E-24	.9807651190E-23	.1010494746E-22	-.1568629523E-24
9	.1453150560E-27	.6370257555E-27	.3411953202E-26	-.1813957874E-27
10	.2064648248E-29	-.1815198037E-28	.1860298951E-28	-.1882078057E-29

A= 1.0 B= 1.3

N	T(N)	U(N)	V(N)	W(N)
1	-.3939967854E-02	.3784446421E-01	.4131786777E-01	-.3967190971E-02
2	-.1773856271E-04	.2886918099E-03	.3043451020E-03	-.1782613712E-04
3	-.3802158789E-07	.8723323995E-06	.9059025921E-06	-.3816762127E-07
4	-.4753525182E-10	.1407555373E-08	.1449539970E-08	-.4768465884E-10
5	-.3889714414E-13	.1411116853E-11	.1445480467E-11	-.3900064621E-13
6	-.2244192764E-16	.9638034307E-15	.9836413282E-15	-.2249502114E-16
7	-.9600665307E-20	.4771060570E-18	.4858285598E-18	-.9654441921E-20
8	-.2829065157E-23	.1766514361E-21	.1843818244E-21	-.3502496424E-23
9	.4618226638E-26	.9738237773E-26	.9578778859E-25	-.5739294370E-26
10	.6836298754E-28	-.5871494499E-27	.6039654659E-27	-.6237747980E-28

TABLE 3. COEFFICIENTS FOR THE ASCENDING SERIES FOR THE CROSS PRODUCTS OF MODIFIED BESSEL FUNCTIONS FOR WALL RATIOS B = 1.4 AND B = 1.5

A= 1.0 B= 1.4

N	T(N)	U(N)	V(N)	W(N)
1	-.8989455100E-02	.6410199421E-01	.7176388169E-01	-.9091860040E-02
2	-.7197368649E-04	.8733442750E-03	.9347872390E-03	-.7255964327E-04
3	-.2743086354E-06	.4700586304E-05	.4934969013E-05	-.2760462893E-06
4	-.6097492287E-09	.1349822503E-07	.1401952121E-07	-.6129104664E-09
5	-.8870841100E-12	.2407384538E-10	.2483254243E-10	-.8909761686E-12
6	-.9099700935E-15	.2924513211E-13	.3002354384E-13	-.9134163756E-15
7	-.6936001285E-18	.2575298253E-16	.2634377374E-16	-.6955187941E-18
8	-.4125146756E-21	.1722008661E-19	.1750788645E-19	-.4050530232E-21
9	-.2681943668E-24	.9581529036E-23	.8586642047E-23	-.1219884915E-24
10	-.1130718874E-26	.1267418110E-25	-.4870523935E-26	.8860415732E-27

A= 1.0 B= 1.5

N	T(N)	U(N)	V(N)	W(N)
1	-.1694040034E-01	.9576549775E-01	.1097674459E+00	-.1722141779E-01
2	-.2120029825E-03	.2047295679E-02	.2222931588E-02	-.2145169505E-03
3	-.1262745597E-05	.1724792053E-04	.1829540722E-04	-.1274398787E-05
4	-.4386356689E-08	.7746495100E-07	.8110657298E-07	-.4419490848E-08
5	-.9971866976E-11	.2160044186E-09	.2242879582E-09	-.1003562656E-10
6	-.1598394448E-13	.4101811590E-12	.4234644048E-12	-.1607253201E-13
7	-.1903175257E-16	.5645274785E-15	.5803466818E-15	-.1912453401E-16
8	-.1749841326E-19	.5889434271E-18	.6034405285E-18	-.1756763143E-19
9	-.1286095309E-22	.4822215810E-21	.4918352158E-21	-.1277656214E-22
10	-.8671027869E-26	.3257433753E-24	.3151945055E-24	-.6684017409E-26

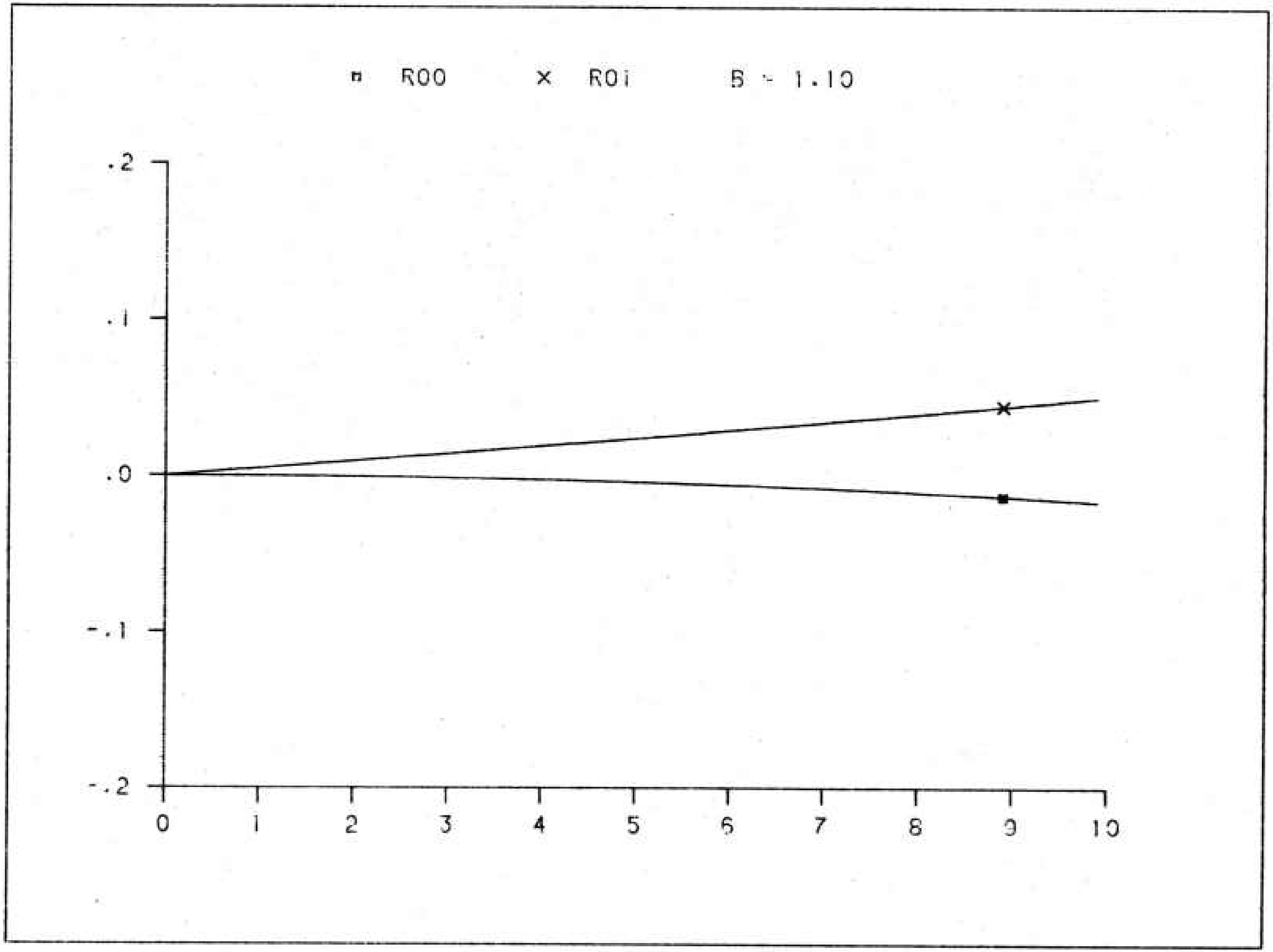

Figure 1. Graph of $R_{0,0}(s)$ and $R_{0,1}(s)$ for Wall Ratio = 1.1

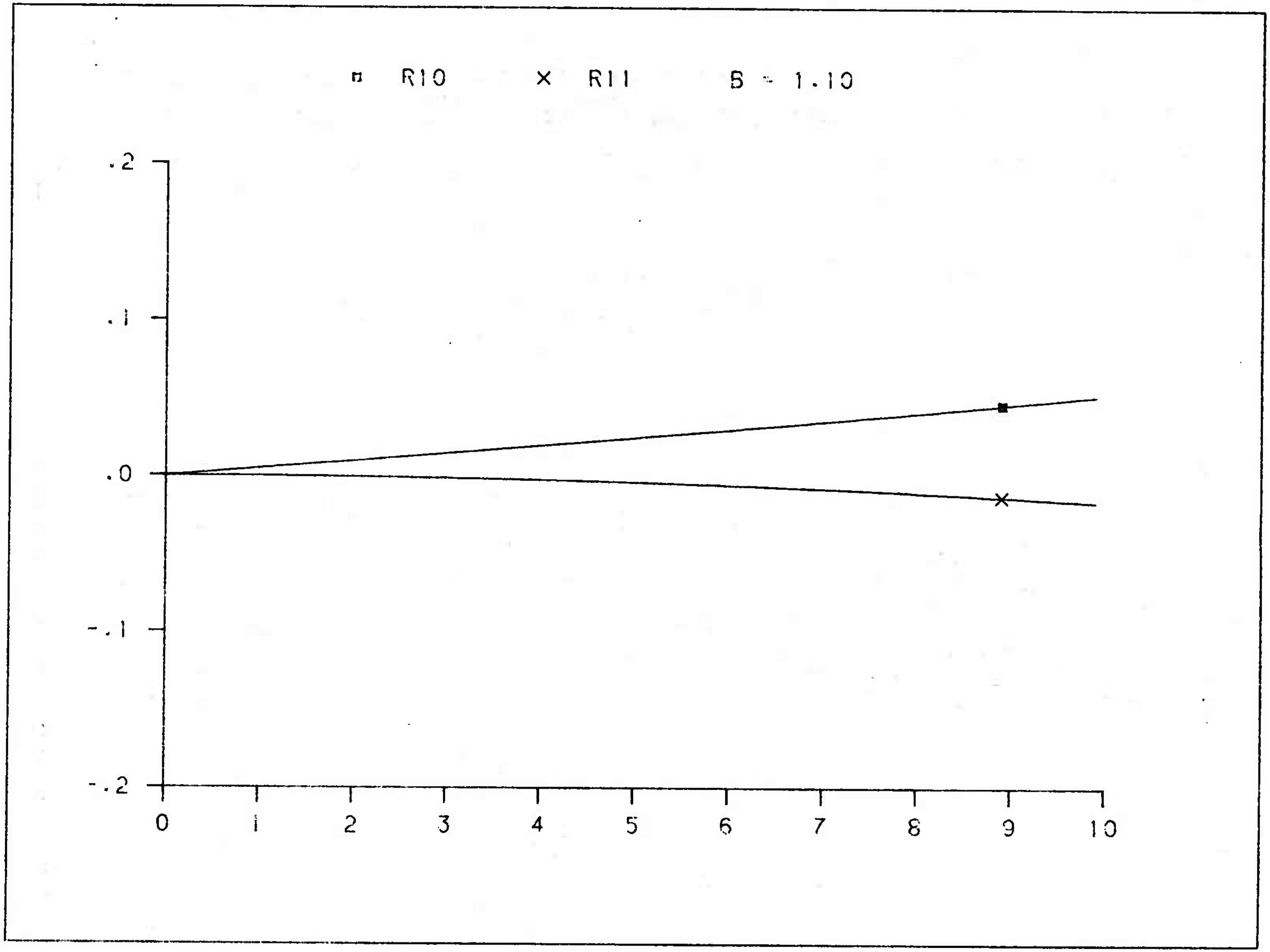

Figure 2. Graph of $R_{1,0}(s)$ and $R_{1,1}(s)$ for Wall Ratio = 1.1

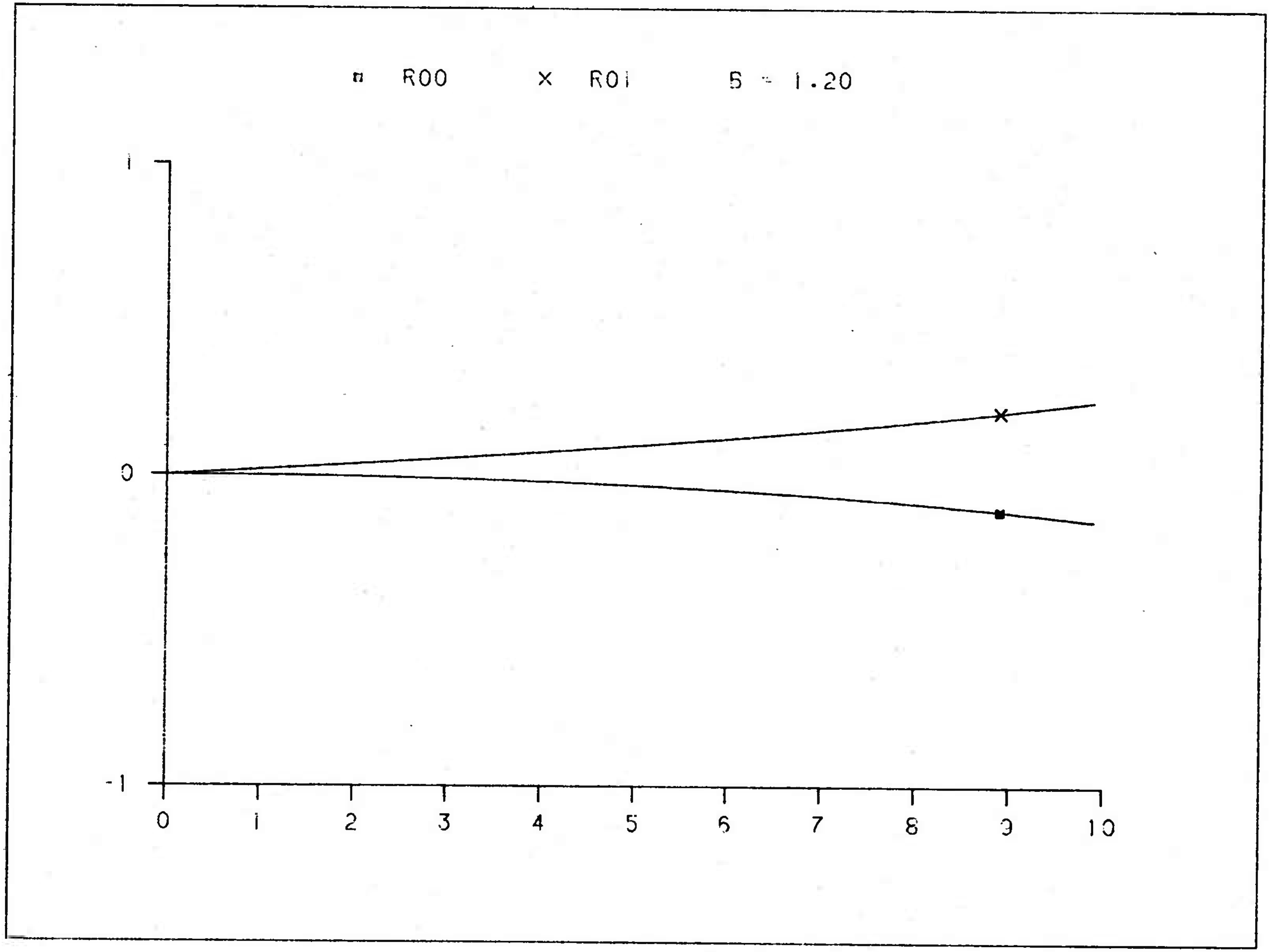

Figure 3. Graph of $R_{0,0}(s)$ and $R_{0,1}(s)$ for Wall Ratio = 1.2

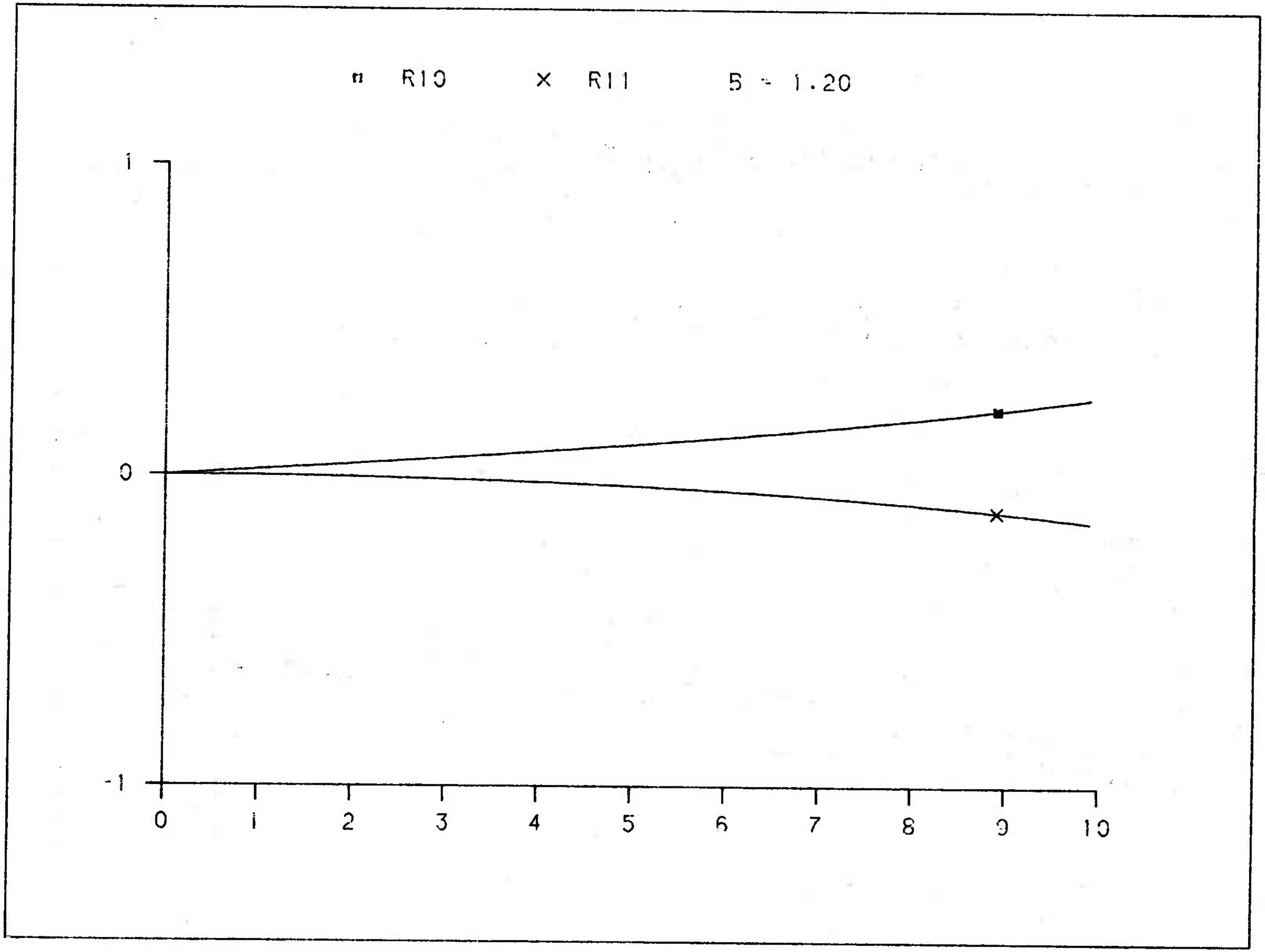

Figure 4. Graph of $R_{1,0}(s)$ and $R_{1,1}(s)$ for Wall Ratio = 1.2

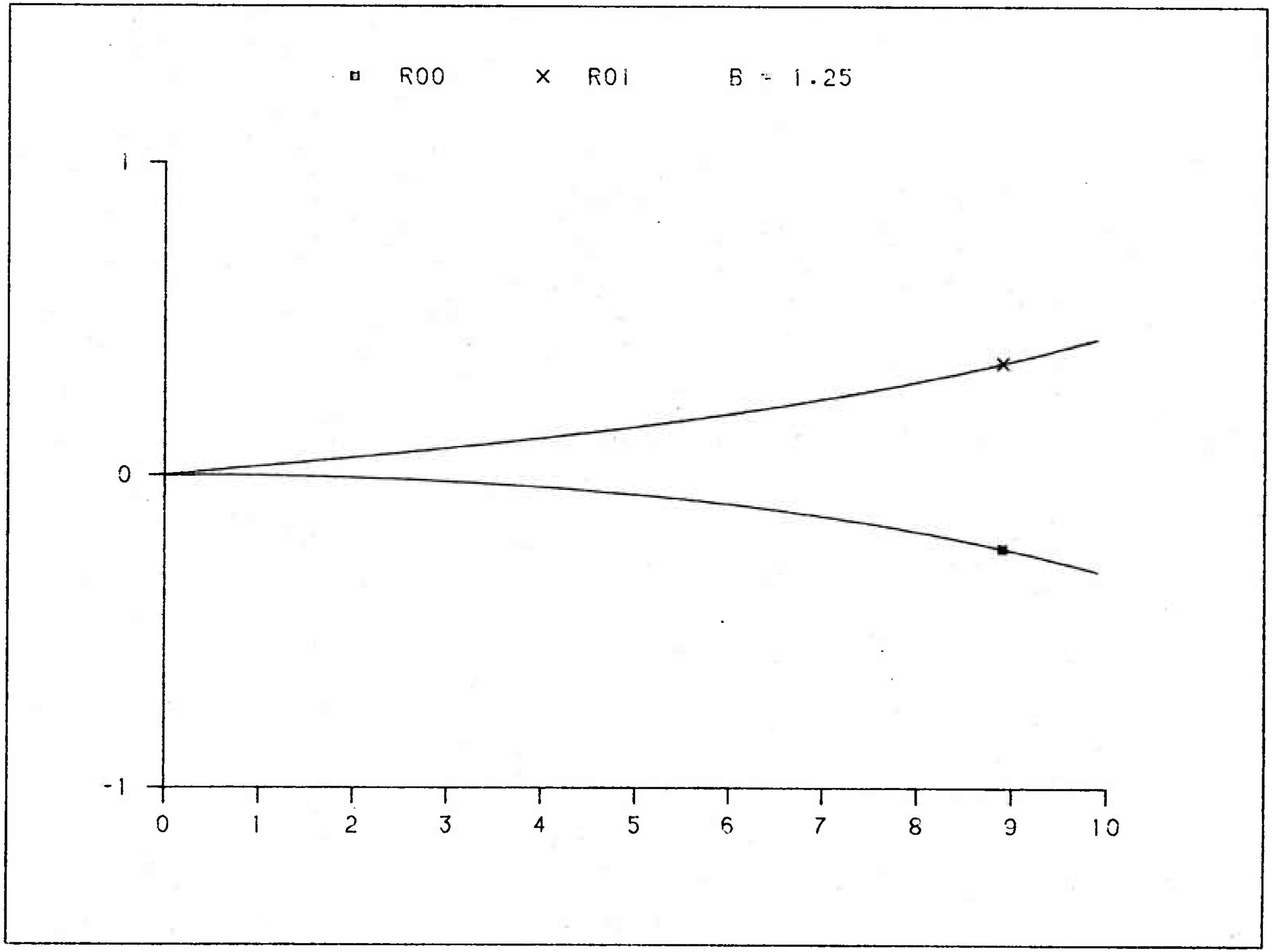

Figure 5. Graph of $R_{0,0}(s)$ and $R_{0,1}(s)$ for Wall Ratio = 1.25

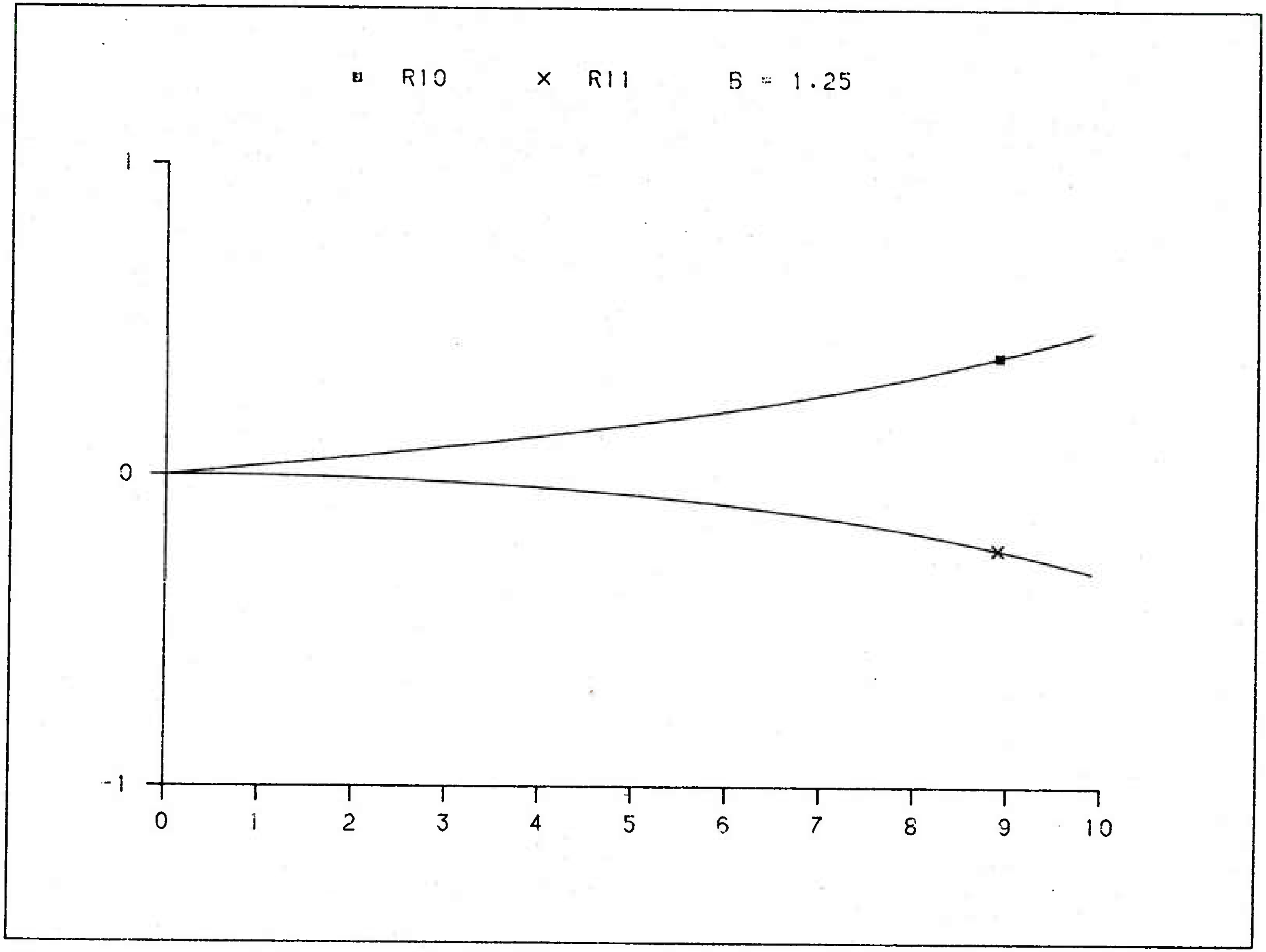

Figure 6. Graph of $R_{1,0}(s)$ and $R_{1,1}(s)$ for Wall Ratio = 1.25

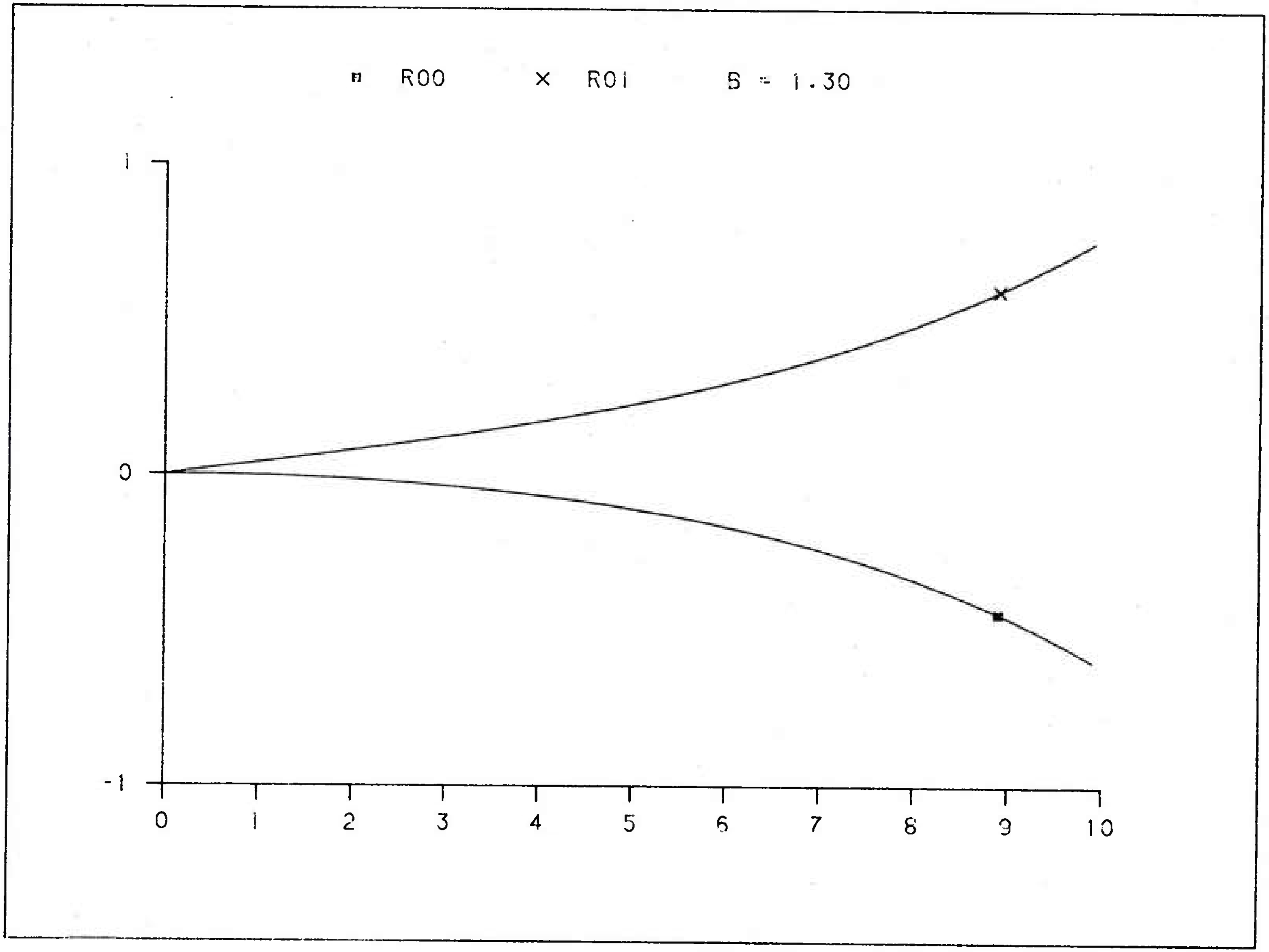

Figure 7. Graph of $R_{0,0}(s)$ and $R_{0,1}(s)$ for Wall Ratio = 1.3

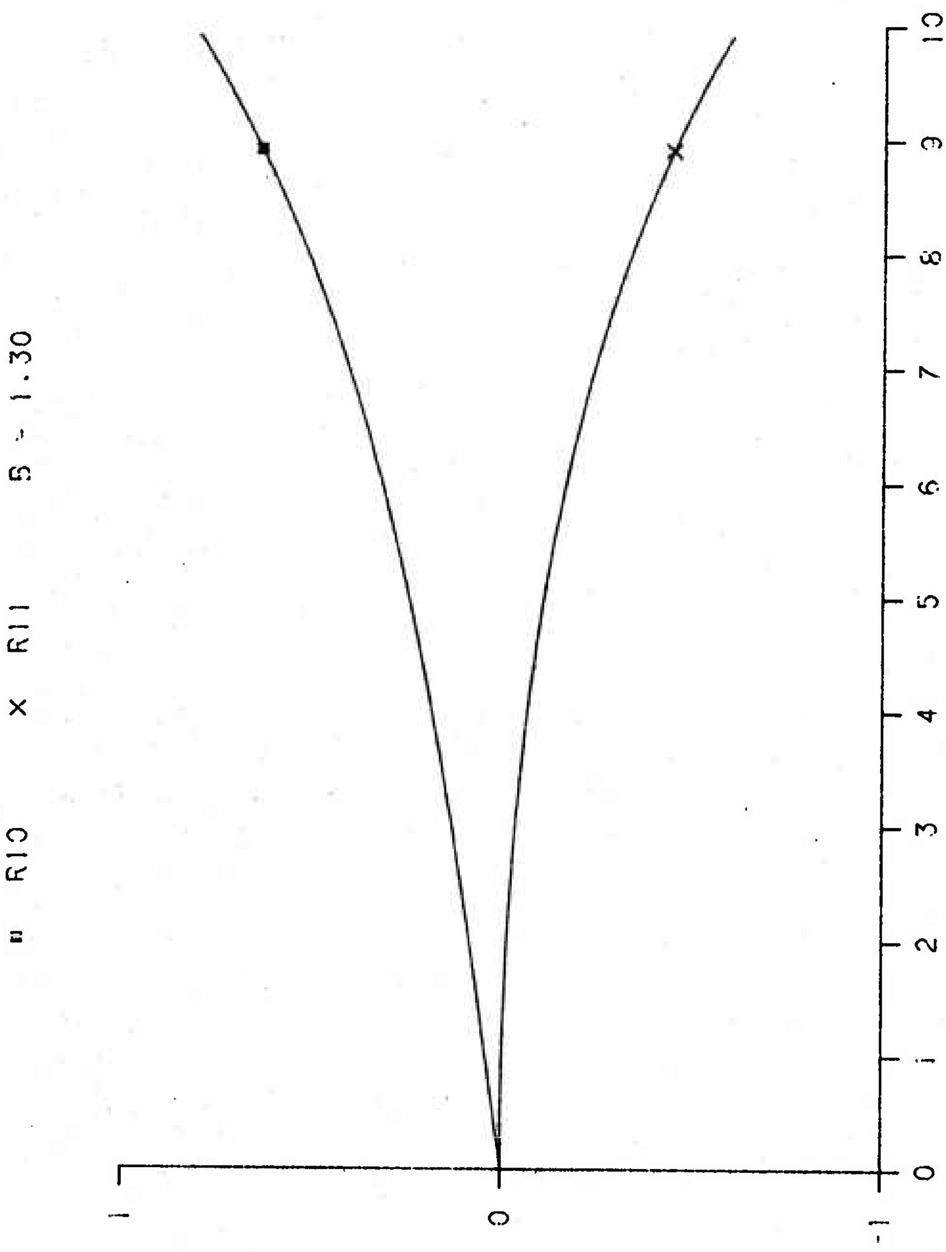

Figure 8. Graph of $R_{1,0}(s)$ and $R_{1,1}(s)$ for Wall Ratio = 1.3

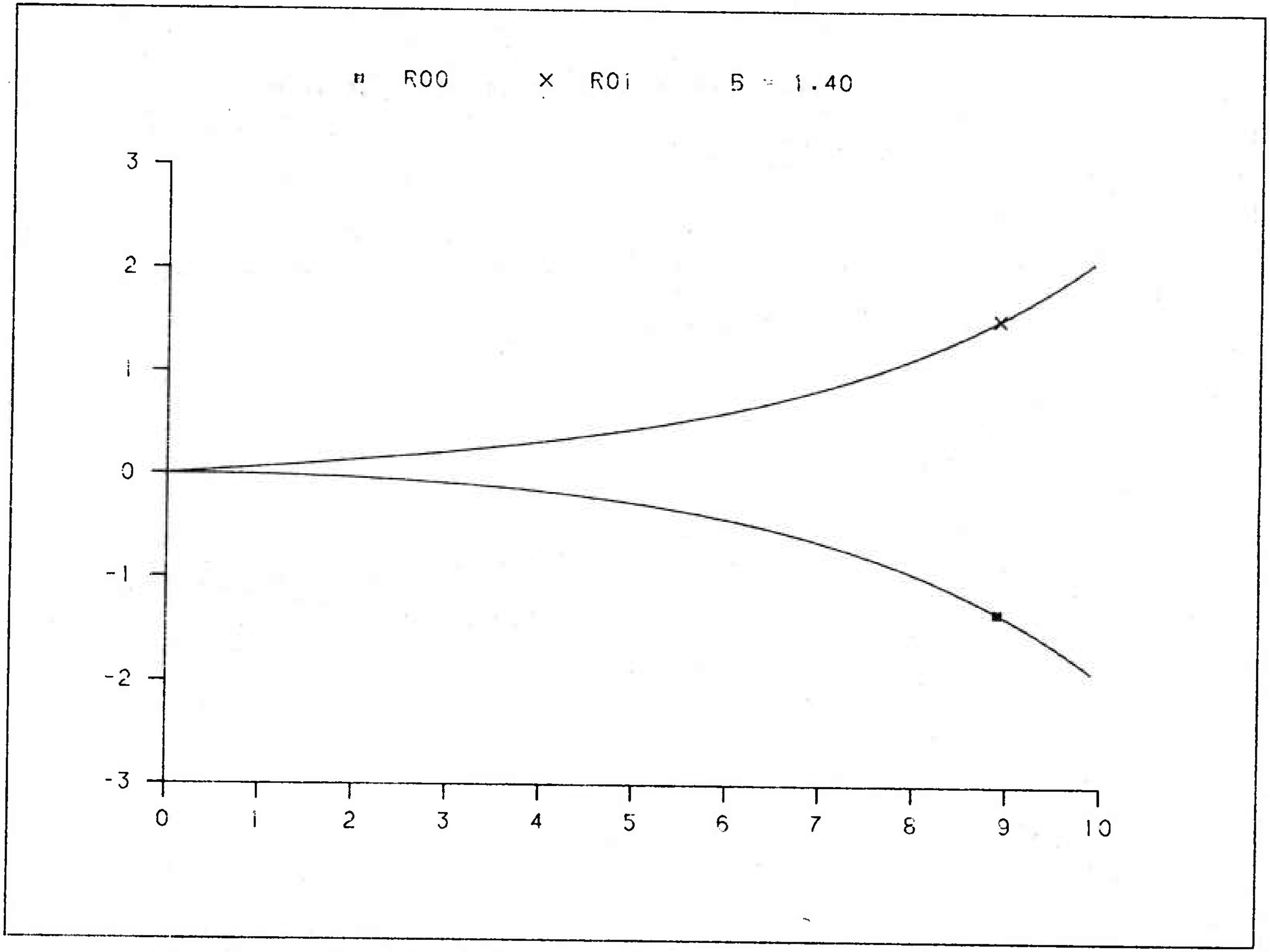

Figure 9. Graph of $R_{0,0}(s)$ and $R_{0,1}(s)$ for Wall Ratio = 1.4

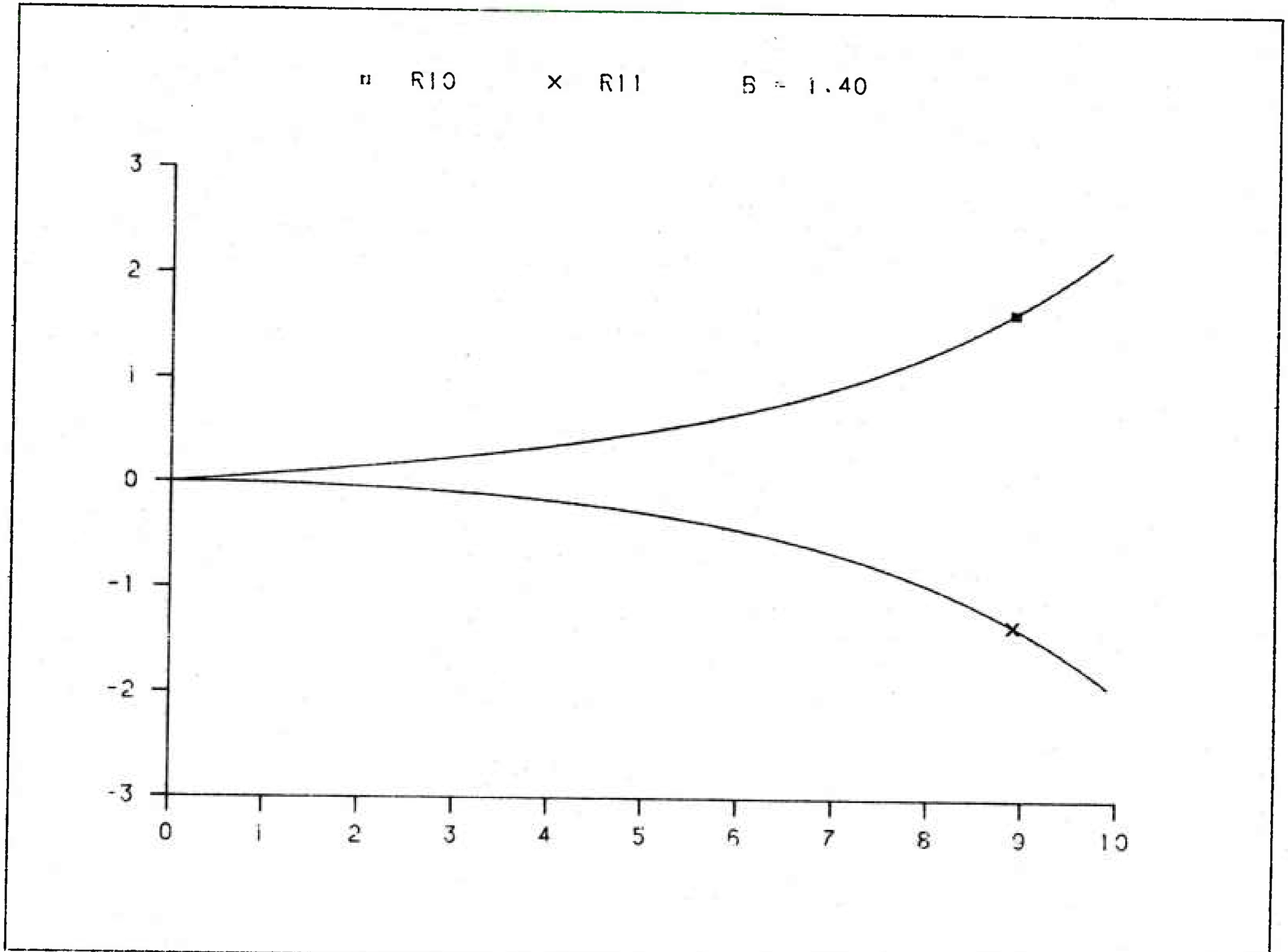

Figure 10. Graph of $R_{1,0}(s)$ and $R_{1,1}(s)$ for Wall Ratio = 1.4

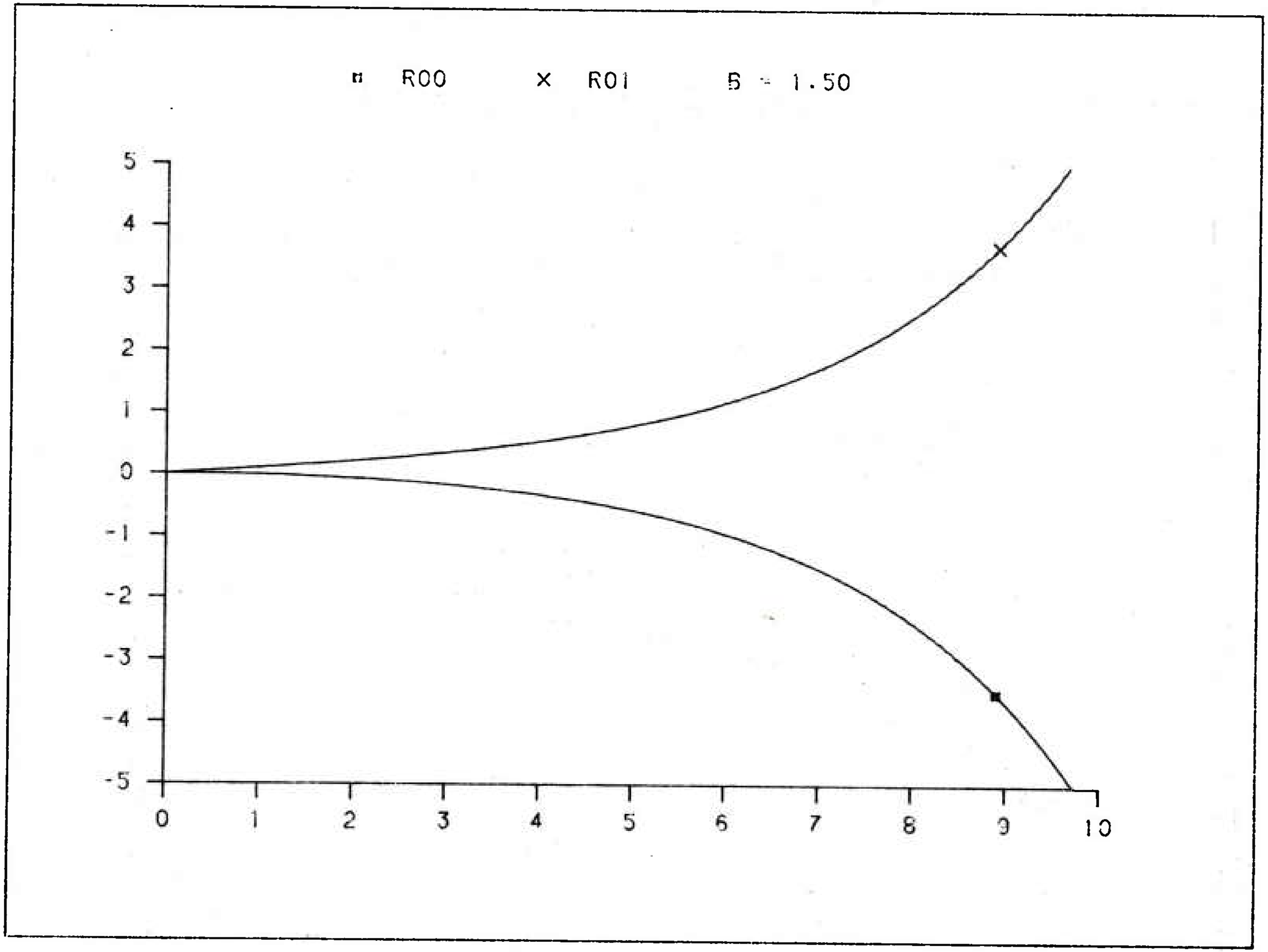

Figure 11. Graph of $R_{0,0}(s)$ and $R_{0,1}(s)$ for Wall Ratio = 1.5

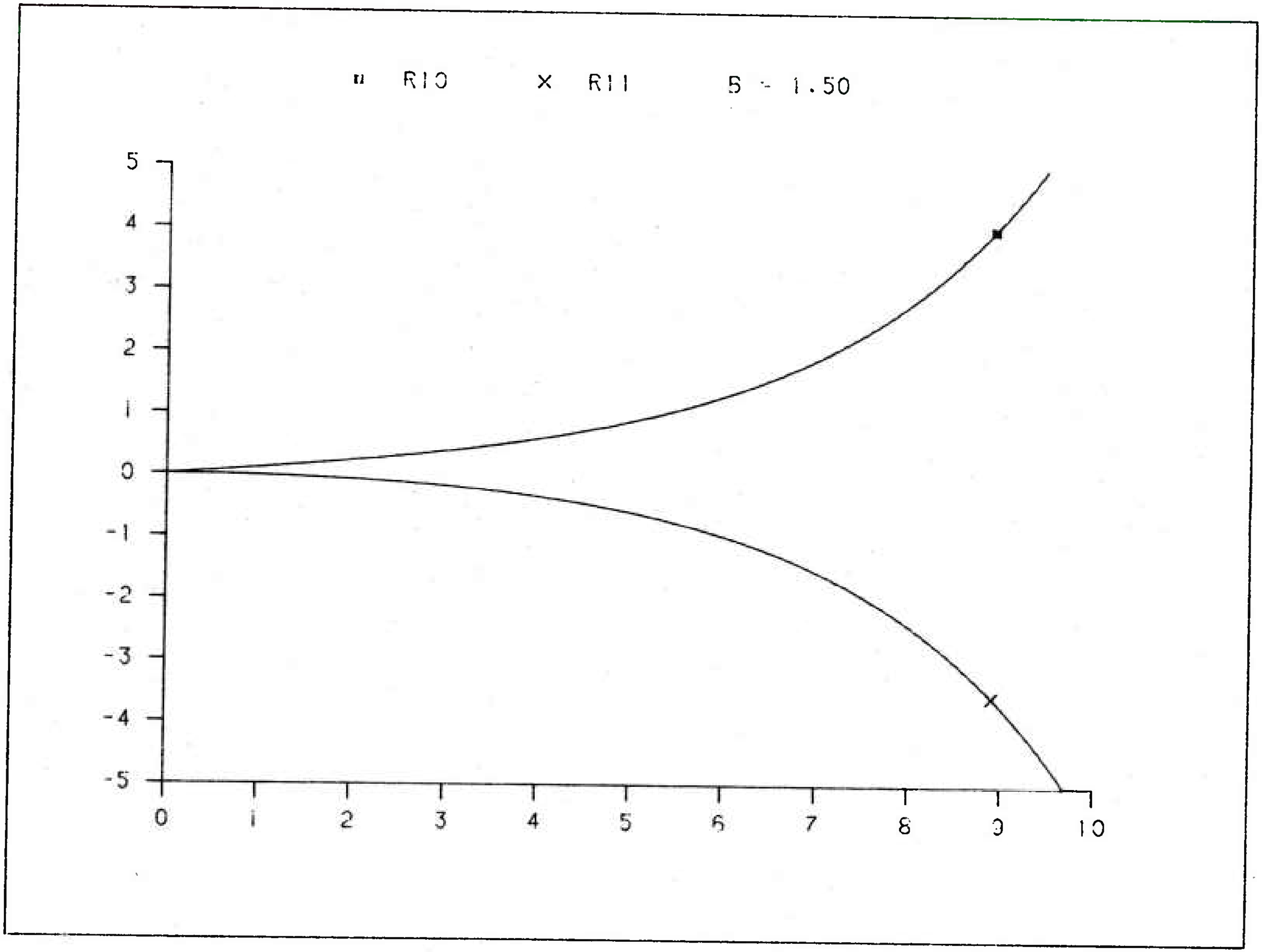

Figure 12. Graph of $R_{1,0}(s)$ and $R_{1,1}(s)$ for Wall Ratio = 1.5

$$I_0(sa) = \left[e^{sa}P_0(sa) + ie^{-sa}Q_0(sa)\right] \Big/ \sqrt{2\pi sa} \quad , \tag{53}$$

$$K_0(sa) = \pi e^{-sa}Q_0(sa)/\sqrt{2\pi sa} \quad , \tag{54}$$

$$I_1(sa) = \left[e^{sa}P_1(sa) - i\, e^{-sa}Q_1(sa)\right] \Big/ \sqrt{2\pi sa} \quad , \tag{55}$$

$$K_1(sa) = \pi e^{-sa}\, Q_1(sa) \,/\, \sqrt{2\pi sa} \quad , \tag{56}$$

where

$$P_0(sa) = 1 + \frac{1^2}{1!(8sa)} + \frac{1^2 \cdot 3^2}{2!(8sa)^2} + \cdots \quad , \tag{57}$$

$$Q_0(sa) = 1 - \frac{1^2}{1!(8sa)} + \frac{1^2 \cdot 3^2}{2!(8sa)^2} - \cdots \quad , \tag{58}$$

$$P_1(sa) = 1 - \frac{1 \cdot 3}{1!(8sa)} - \frac{1^2 \cdot 3 \cdot 5}{2!(8sa)^2} - \cdots \quad , \tag{59}$$

$$Q_1(sa) = 1 + \frac{1 \cdot 3}{1!(8sa)} - \frac{1^2 \cdot 3 \cdot 5}{2!(8sa)^2} + \cdots \quad . \tag{60}$$

Similar formulas for $P_0(sb)$, $Q_0(sb)$, $P_1(sb)$, and $Q_1(sb)$ are obtained by substituting b for a in the preceding equations.

The following formulas for the $L_{i,j}(s)$ functions are obtained by successive substitutions.

$$L_{0,0}(s) = \left[e^{-s(b-a)}P_0(sa)Q_0(sb) - e^{s(b-a)}P_0(sb)Q_0(sa)\right] \Big/ \sqrt{4abs^2} \tag{61}$$

$$L_{0,1}(s) = \left[e^{-s(b-a)}P_0(sa)Q_1(sb) + e^{s(b-a)}P_1(sb)Q_0(sa)\right] \Big/ \sqrt{4abs^2} \tag{62}$$

$$L_{1,0}(s) = \left[e^{-s(b-a)}P_1(sa)Q_0(sb) + e^{s(b-a)}P_0(sb)Q_1(sa)\right] \Big/ \sqrt{4abs^2} \tag{63}$$

$$L_{1,1}(s) = \left[e^{-s(b-a)}P_1(sa)Q_1(sb) - e^{s(b-a)}P_1(sb)Q_1(sa)\right] \Big/ \sqrt{4abs^2} \tag{64}$$

Next, we eliminate the exponential functions from the subsequent analysis. Assume

$$L_{i,j}(s) = U_{i,j}(s) - (-1)^{i+j} V_{i,j}(s) \quad ; \tag{65}$$

then $U_{i,j}(s)$ and $V_{i,j}(s)$ will also satisfy Eqs. (2). Finally we write

$$U_{i,j}(s) = e^{-s(b-a)} \; E_{i,j}(s) \quad , \tag{66}$$

and

$$V_{i,j}(s) = e^{s(b-a)} \; F_{i,j}(s) \quad . \tag{67}$$

On substituting these expressions for $U_{i,j}(s)$ and $V_{i,j}(s)$ we obtain the following sets of differential equations for $E_{i,j}(s)$ and $F_{i,j}(s)$.

$$E'_{0,0} = a\,E_{1,0} - bE_{0,1} + (b-a)\,E_{0,0} \quad , \tag{68}$$

$$E'_{0,1} = a\,E_{1,1} - bE_{0,0} + \left[(b-a) - \frac{1}{s}\right] E_{0,1} \quad , \tag{69}$$

$$E'_{1,0} = a\,E_{0,0} - bE_{1,1} + \left[(b-a) - \frac{1}{s}\right] E_{1,0} \quad , \tag{70}$$

$$E'_{1,1} = a\,E_{0,1} - bE_{1,0} + \left[(b-a) - \frac{2}{s}\right] E_{1,1} \quad , \tag{71}$$

and

$$F'_{0,0} = a\,F_{1,0} - bF_{0,1} - (b-a)\,F_{0,0} \quad , \tag{72}$$

$$F'_{0,1} = a\,F_{1,1} - bF_{0,0} - \left[(b-a) - \frac{1}{s}\right] F_{0,1} \quad , \tag{73}$$

$$F'_{1,0} = a\,F_{0,0} - bF_{1,1} - \left[(b-a) - \frac{1}{s}\right] F_{1,0} \quad , \tag{74}$$

$$F'_{1,1} = a\,F_{0,1} - bF_{1,0} - \left[(b-a) - \frac{2}{s}\right] F_{1,1} \quad . \tag{75}$$

We assume*

$$E_{0,0} = \sum_{n=1}^{\infty} t_n\, s^{-n} \quad , \tag{76}$$

$$E_{0,1} = \sum_{n=1}^{\infty} u_n\, s^{-n} \quad , \tag{77}$$

$$E_{1,0} = \sum_{n=1}^{\infty} v_n\, s^{-n} \quad , \tag{78}$$

$$E_{1,1} = \sum_{n=1}^{\infty} w_n\, s^{-n} \quad . \tag{79}$$

We can prove that

$$F_{0,0} = \sum_{n=1}^{\infty} (-1)^{n+1} t_n s^{-n} \quad , \tag{80}$$

$$F_{0,1} = \sum_{n=1}^{\infty} (-1)^{n+1} u_n s^{-n} \quad , \tag{81}$$

$$F_{1,0} = \sum_{n=1}^{\infty} (-1)^{n+1} v_n s^{-n} \quad , \tag{82}$$

The coefficients t_i, u_i, v_i, w_i are new and are not the same as in Eqs. (32)-(35).

$$F_{1,1} = \sum_{n=1}^{\infty} (-1)^{n+1} w_n s^{-n} \quad , \tag{83}$$

so a separate derivation for the $F_{i,j}(s)$ functions is not required. The initial terms are

$$t_1 = 1/\sqrt{4ab} \quad , \tag{84}$$

$$u_1 = 1/\sqrt{4ab} \quad , \tag{85}$$

$$v_1 = 1/\sqrt{4ab} \quad , \tag{86}$$

and

$$w_1 = 1/\sqrt{4ab} \quad . \tag{87}$$

The series expansions for $E_{0,0}$, $E_{0,1}$, $E_{1,0}$, and $E_{1,1}$ are substituted in Eqs. (68) - (71) and the coefficients of each power of s set equal to zero. We examine the coefficients of $1/s$, $1/s^2$, and $1/s^3$ in detail. We find from terms in $1/s$

$$av_1 - bu_1 + (b-a)t_1 = 0 \quad , \tag{88}$$

$$aw_1 - bt_1 + (b-a)u_1 = 0 \quad , \tag{89}$$

$$at_1 - bw_1 + (b-a)v_1 = 0 \quad , \tag{90}$$

$$au_1 - bv_1 + (b-a)w_1 = 0 \quad . \tag{91}$$

We see these equations are consistent with the initial conditions given in Eqs. (84) - (87).

The coefficients of $1/s^2$ lead to the following equations:

$$av_2 - bu_2 + (b-a)t_2 = -t_1 \quad , \tag{92}$$

$$aw_2 - bt_2 + (b-a)u_2 = 0 \quad , \tag{93}$$

$$at_2 - bw_2 + (b-a)v_2 = 0 \quad , \tag{94}$$

$$au_2 - bv_2 + (b-a)w_2 = w_1 \quad . \tag{95}$$

We find these equations are not linearly independent, and must be supplemented by equations obtained from the coefficients of $1/s^3$.

$$(b-a)t_3 \quad -bu_3 \quad +av_3 \quad = -2t_2 \tag{96}$$

$$-bt_3 \quad +(b-a)u_3 \quad +aw_3 \quad = -u_2 \tag{97}$$

$$at_3 \quad +(b-a)v_3 \quad -bw_3 \quad = -v_2 \tag{98}$$

$$au_3 \quad -bv_3 \quad +(b-a)w_3 = 0 \tag{99}$$

Consequently,

$$2t_2 + u_2 + v_2 = 0 \quad . \tag{100}$$

We now solve Eqs. (92), (93), (94), and (100), obtaining

$$t_2 = t_1(b-a)/8ab \quad , \tag{101}$$

$$u_2 = t_1(3a+b)/8ab \quad , \tag{102}$$

$$v_2 = - t_1(a+3b)/8ab \quad , \tag{103}$$

$$w_2 = t_1(3a-3b)/8ab \quad . \tag{104}$$

We proceed with the general case in a similar manner, drawing on coefficients from $1/s^n$ and $1/s^{n+1}$ for a set of linearly independent equations.

$$(b-a)t_n - bu_n + av_n = -(n-1)t_{n-1} \tag{105}$$

$$- b\, t_n + (b-a)u_n + aw_n = -(n-2)u_{n-1} \tag{106}$$

$$a\, t_n + (b-a)v_n - b\, w_n = -(n-2)v_{n-1} \tag{107}$$

$$a\, u_n - b\, v_n + (b-a)w_n = -(n-3)w_{n-1} \tag{108}$$

On adding, we find

$$(n-1)t_{n-1} + (n-2)u_{n-1} + (n-2)v_{n-1} + (n-3)w_{n-1} = 0 \quad . \tag{109}$$

Now replace n by (n+1). We find

$$n\, t_n + (n-1)u_n + (n-1)v_n + (n-2)w_n = 0 \quad . \tag{110}$$

To simplify, add Eq. (105) to Eq. (106), (107) and (108) in turn. We find

$$- at_n - au_n + av_n + aw_n = x_{n-1} \quad , \tag{111}$$

$$bt_n - bu_n + bv_n - bw_n = y_{n-1} \quad , \tag{112}$$

$$(b-a)t_n - (b-a)u_n - (b-a)v_n + (b-a)w_n = z_{n-1} \quad , \tag{113}$$

where

$$x_{n-1} = -(n-1)t_{n-1} - (n-2)u_{n-1} \quad , \tag{114}$$

$$y_{n-1} = -(n-1)t_{n-1} - (n-2)v_{n-1} \quad , \tag{115}$$

$$z_{n-1} = -(n-1)t_{n-1} - (n-3)w_{n-1} \quad . \tag{116}$$

We now solve Eqs. (110) - (113) algebraically by Cramer's rule

$$\begin{aligned}\Delta t_n = &[(4n-6)a^2 - (12n-16)ab + (4n-6)b^2][n-1]t_{n-1} \\ &+ [(4n-6)(-ab+b^2)][n-2]u_{n-1} \\ &+ [(4n-6)(a^2-ab)][n-2]v_{n-1} + [(-4n-4)(ab)][n-3]w_{n-1} \quad ,\end{aligned} \tag{117}$$

$$\begin{aligned}\Delta u_n = &[-(4n-6)(ab-b^2)][n-1]t_{n-1} \\ &+ [(4n-2)a^2 - (12n-12)ab + (4n-6)b^2][n-2]u_{n-1} \\ &+ [-(4n-4)ab][n-2]v_{n-1} \\ &+ [(4n-2)(a^2-ab)][n-3]w_{n-1} \quad ,\end{aligned} \tag{118}$$

$$\Delta v_n = [(4n-6)(a^2-ab)][n-1]t_{n-1} + [-(4n-4)ab][n-2]u_{n-1}$$
$$+ [(4n-6)a^2-(12n-12)ab+(4n-2)b^2][n-2]v_{n-1} \tag{119}$$
$$+ [-(4n-2)(ab-b^2)][n-3]w_{n-1} \quad ,$$

$$\Delta w_n = [-(4n-4)ab][n-1]t_{n-1} + [(4n-2)(a^2-ab)][n-2]u_{n-1}$$
$$+ [-(4n-2)(a^2-ab)][n-2]v_{n-1} \tag{120}$$
$$+[(4n-2)a^2-(12n-8)ab+(4n-2)b^2][n-3]w_{n-1} \quad ,$$

where

$$\Delta = 16(n-1)ab\ (b-a),\ n>1. \tag{121}$$

These formulas were used in an algorithm for the $L_{i,j}$ (s) functions which has been incorporated in the thick-walled cylinder problem. The variable s was taken sufficiently large, so the asymptotic series could be truncated when a given term was sufficiently small, well before the smallest term of the asymptotic series. Although the formulas are cumbersome, computations were extremely rapid.

The coefficients are listed for various wall ratios in Tables 4 thru 9 and graphs of the $L_{i,j}(s)$ given in Eqs. 61 thru 64 for positive real s are shown in Figures 13 thru 24.

V. ASYMPTOTIC SERIES FOR SQUARES AND PRODUCTS OF MODIFIED BESSEL FUNCTIONS

The asymptotic series for the $L_{i,j}(s)$ functions eliminated exponential overrun for all wall ratios of interest, but cancellation of leading terms still occurred, and led to serious computational difficulties for thin-walled cylinders when the arguments for the Bessel functions were large. New dependent variables were selected in which the differences of nearly equal numbers were obtained by analysis, thus bypassing the numerical problem. These new variables were suggested by the Laplace

TABLE 4. COEFFICIENTS FOR ASYMPTOTIC EXPANSIONS OF CROSS PRODUCTS OF MODIFIED BESSEL FUNCTIONS FOR WALL RATIO B= 1.10

N	T(N)	U(N)	V(N)	W(N)
1	.47673129462E+00	.47673129462E+00	.47673129462E+00	.47673129462E+00
2	.54174010753E-02	.22211344409E+00	-.23294824624E+00	-.16252203226E-01
3	.54451036944E-01	.76643912940E-02	-.78490754215E-02	-.16298374258E+00
4	.83370192813E-02	.77299751461E-01	-.79157086153E-01	-.13888036525E-01
5	.84685521683E-01	.19762277820E-01	-.20078123691E-01	-.14072140831E+00
6	.39716166356E-01	.20199449906E+00	-.20520527609E+00	-.55560778197E-01
7	.41003733353E+00	.13224794957E+00	-.13382798690E+00	-.57215622333E+00
8	.39053316892E+00	.13776130998E+01	-.13939402435E+01	-.50166252306E+00
9	.41200193058E+01	.16763260064E+01	-.16923086167E+01	-.52789030111E+01
10	.65486591945E+01	.17888719975E+02	-.18057449980E+02	-.79959569865E+01
11	.70942622349E+02	.34431546678E+02	-.34706052182E+02	-.86404658353E+02
12	.16757043407E+03	.37821301119E+03	-.38109652267E+03	-.19781382402E+03
13	.18718220218E+04	.10411899629E+04	-.10524507758E+04	-.22052108098E+04
14	.61042051319E+04	.11966942598E+05	-.11796290579E+05	-.69910387644E+04
15	.69068346015E+05	.35686838313E+05	-.52096376381E+05	-.82217536258E+05
16	.38574440625E+06	.10791256873E+07	.42428941395E+05	-.24884353675E+06
17	-.28042279337E+07	-.39620561105E+08	-.44291383874E+08	-.10431624822E+08
18	.51181556191E+09	.33760099268E+10	.33006707587E+10	.47264076138E+09
19	-.40182635186E+11	-.28274268478E+12	-.28183232019E+12	-.40749172995E+11
20	.35073702576E+13	.25329867710E+14	.25218072761E+14	.35098498136E+13
21	-.32062221964E+15	-.23934506795E+16	-.23837753433E+16	-.32117296745E+15
22	.30834041010E+17	.23809405050E+18	.23716791311E+18	.30882085289E+17
23	-.31095369792E+19	-.24869683002E+20	-.24776781593E+20	-.31143744765E+19
24	.32809400425E+21	.27214935329E+22	.27117067227E+22	.32860121853E+21
25	-.36140421306E+23	-.31135848593E+24	-.31027874413E+24	-.36196193904E+23
26	.41477268633E+25	.37171362745E+26	.37046854572E+26	.41541378090E+25
27	-.49503895324E+27	-.46227006815E+28	-.46077223465E+28	-.49580793934E+27
28	.61337097044E+29	.59789781408E+30	.59602120631E+30	.61433180821E+29
29	-.78767677474E+31	-.80308030582E+32	-.80063551528E+32	-.78892537019E+31
30	.10467353494E+34	.11186533249E+35	.11153463335E+35	.10484203123E+34

TABLE 4. COEFFICIENTS FOR ASYMPTOTIC EXPANSIONS OF CROSS PRODUCTS OF MODIFIED BESSEL FUNCTIONS FOR WALL RATIO B= 1.10 (CONT'D)

N	T(N)	U(N)	V(N)	W(N)
31	-.14372839648E+36	-.16139206283E+37	-.16092823317E+37	-.14396420498E+36
32	.20362958970E+38	.24088076792E+39	.24020706303E+39	.20397139165E+38
33	-.29725120266E+40	-.37151168111E+41	-.37049950723E+41	-.29776373183E+40
34	.44646890377E+42	.59148124487E+43	.58991002644E+43	.44726306472E+42
35	-.68903995954E+44	-.97114314528E+45	-.96862566609E+45	-.69031021747E+44
36	.10911248216E+47	.16428588069E+48	.16386994870E+48	.10932201117E+47
37	-.17703165196E+49	-.28609880740E+50	-.28539084524E+50	-.17738774711E+49
38	.29383682581E+51	.51247885559E+52	.51123847203E+52	.29445981639E+51
39	-.49809742682E+53	-.94350602114E+54	-.94127086419E+54	-.49921850407E+53
40	.86071820393E+55	.17840410026E+57	.17799016434E+57	.86279165021E+55
41	-.15128659836E+58	-.34622268580E+59	-.34543542986E+59	-.15168045272E+58
42	.26976677031E+60	.68914575756E+61	.68760916952E+61	.27053459683E+60
43	-.48637915911E+62	-.14060474650E+64	-.14029715548E+64	-.48791444523E+62
44	.88271696835E+64	.29387574810E+66	.29324465346E+66	.88586357609E+64
45	-.16023879745E+67	-.62886307143E+68	-.62753670693E+68	-.16089943670E+67
46	.28811574714E+69	.13770257138E+71	.13741718463E+71	.28953581204E+69
47	-.50465473014E+71	-.30838781977E+73	-.30775951060E+73	-.50777821203E+71
48	.83341210357E+73	.70600382511E+75	.70458914479E+75	.84043849350E+73
49	-.11963629615E+76	-.16514531332E+78	-.16481972041E+78	-.12125204298E+76
50	.10587104508E+78	.39452829107E+80	.39376266120E+80	.10966731235E+78
51	.18065997824E+80	-.96217528225E+82	-.96033665282E+82	.17155065965E+80
52	-.15417678053E+83	.23944923525E+85	.23899850869E+85	-.15194540839E+83
53	.66860584768E+85	-.60783004057E+87	-.60670259532E+87	.66302841794E+85
54	-.24480527067E+88	.15732339011E+90	.15703573773E+90	-.24338327474E+88
55	.83509009970E+90	-.41503642002E+92	-.41428813980E+92	.83139354065E+90
56	-.27533338596E+93	.11155942019E+95	.11136102816E+95	-.27435396077E+93
57	.89303358299E+95	-.30542490273E+97	-.30488899048E+97	.89038956437E+95
58	-.28774647705E+98	.85140695235E+99	.84993252235E+99	-.28701948723E+98
59	.92653801553+100	-.24158277867+102	-.24116975735+102	.92450275940+100
60	-.29928907915+103	.69752067102+104	.69634306372+104	-.29870911899+103

TABLE 5. COEFFICIENTS FOR ASYMPTOTIC EXPANSIONS OF CROSS PRODUCTS OF MODIFIED BESSEL FUNCTIONS FOR WALL RATIO B= 1.20

N	T(N)	U(N)	V(N)	W(N)
1	.45643546459E+00	.45643546459E+00	.45643546459E+00	.45643546459E+00
2	.95090721789E-02	.19969051576E+00	-.21870866011E+00	-.28527216537E-01
3	.48436836411E-01	.12777815740E-01	-.13372132752E-01	-.14412187521E+00
4	.13526902807E-01	.65901089417E-01	-.68934582495E-01	-.22503565996E-01
5	.71543264015E-01	.30671071992E-01	-.31610977639E-01	-.11798556583E+00
6	.60027031798E-01	.16578959222E+00	-.17077421577E+00	-.83809768258E-01
7	.33563519610E+00	.19223095373E+00	-.19661208551E+00	-.46463191654E+00
8	.55400342480E+00	.11071261757E+01	-.11316956471E+01	-.71000684793E+00
9	.33217622061E+01	.22976270791E+01	-.23392273417E+01	-.42232939953E+01
10	.87825344604E+01	.14273723294E+02	-.14523898854E+02	-.10696719992E+02
11	.57042717430E+02	.44800945183E+02	-.45472752597E+02	-.68972436609E+02
12	.21389885190E+03	.30266990747E+03	-.30704580365E+03	-.25186483902E+03
13	.15143102515E+04	.12968514618E+04	-.13134257133E+04	-.17715662291E+04
14	.74383883476E+04	.95709120152E+04	-.96886397468E+04	-.85503554949E+04
15	.57565105173E+05	.52304526367E+05	-.52884300257E+05	-.65803957659E+05
16	.35081550177E+06	.42230285307E+06	-.42690070931E+06	-.39576826423E+06
17	.29627252110E+07	.28092267607E+07	-.28336378912E+07	-.33402867736E+07
18	.21783401631E+08	.24721778111E+08	-.25090903414E+08	-.23784039613E+08
19	.18998218211E+09	.19678924963E+09	-.19255684619E+09	-.23032909293E+09
20	.20941944846E+10	.17255224164E+10	-.20183339461E+10	-.14317150596E+10
21	-.26589069893E+10	.23657004673E+11	-.10194432803E+11	-.38069615346E+11
22	.11286126874E+13	-.18136012205E+12	-.52106504932E+12	.78999008057E+12
23	-.48478203987E+14	.21019501361E+14	.16853223525E+14	-.52099594459E+14
24	.27534437341E+16	-.10666386409E+16	-.10745843098E+16	.27110586222E+16
25	-.15542496363E+18	.64264145713E+17	.62035414693E+17	-.15571064390E+18

TABLE 5. COEFFICIENTS FOR ASYMPTOTIC EXPANSIONS OF CROSS PRODUCTS OF MODIFIED BESSEL FUNCTIONS FOR WALL RATIO B= 1.20 (CONT'D)

N	T(N)	U(N)	V(N)	W(N)
26	.92489477866E+19	-.39298029087E+19	-.38332316142E+19	.92334135659E+19
27	-.57227432859E+21	.25127437237E+21	.24508805698E+21	-.57173931982E+21
28	.36849287669E+23	-.16673379890E+23	-.16292294194E+23	.36810820823E+23
29	-.24643815275E+25	.11479048033E+25	.11230896726E+25	-.24619604876E+25
30	.17095693515E+27	-.81870997738E+26	-.80198524882E+26	.17079314920E+27
31	-.12285725504E+29	.60421385764E+28	.59251525243E+28	-.12274314366E+29
32	.91355287533E+30	-.46088598873E+30	-.45241146427E+30	.91272802229E+30
33	-.70210263161E+32	.36297829483E+32	.35662703532E+32	-.70148588130E+32
34	.55711885827E+34	-.29486415348E+34	-.28994527167E+34	.55664220978E+34
35	-.45598390678E+36	.24683885776E+36	.24290622167E+36	-.45560358688E+36
36	.38459618075E+38	-.21275268281E+38	-.20951014576E+38	.38428318582E+38
37	-.33399251900E+40	.18864615483E+40	.18589148657E+40	-.33372708522E+40
38	.29839293172E+42	-.17194658553E+42	-.16953746044E+42	.29816117920E+42
39	-.27404600726E+44	.16098721365E+44	.15882001364E+44	-.27383785440E+44
40	.25853693707E+46	-.15471691197E+46	-.15271312423E+46	.25834476505E+46
41	-.25037064045E+48	.15252626339E+48	.15062343764E+48	-.25018840769E+48
42	.24872453558E+50	-.15414807925E+50	-.15229352725E+50	.24854716290E+50
43	-.25331163815E+52	.15960913789E+52	.15775524287E+52	-.25313455301E+52
44	.26432174789E+54	-.16922143889E+54	-.16732182634E+54	.26414051453E+54
45	-.28242557050E+56	.18360916181E+56	.18161518778E+56	-.28223555296E+56
46	.30883886849E+58	-.20377395903E+58	-.20163106065E+58	.30863488263E+58
47	-.34545440939E+60	.23120776579E+60	.22885122101E+60	-.34523032061E+60
48	.39506111347E+62	-.26807043331E+62	-.26541999989E+62	.39480932987E+62
49	-.46168496128E+64	.31746093564E+64	.31441367123E+64	-.46139575636E+64
50	.55110820981E+66	-.38382817648E+66	-.38024845159E+66	.55076878166E+66

TABLE 5. COEFFICIENTS FOR ASYMPTOTIC EXPANSIONS OF CROSS PRODUCTS OF MODIFIED BESSEL FUNCTIONS FOR WALL RATIO B= 1.20 (CONT'D)

N	T(N)	U(N)	V(N)	W(N)
51	-.67165719626E+68	.47359445089E+68	.46929969636E+68	-.67125032613E+68
52	.83540230498E+70	-.59610778229E+70	-.59084776361E+70	.83490440793E+70
53	-.10599994047E+73	.76510927491E+72	.75853549460E+72	-.10593776532E+73
54	.13715417910E+75	-.10010161955E+75	-.99263611958E+74	.13707498134E+75
55	-.18090219589E+77	.13345112360E+77	.13236189693E+77	-.18079933285E+77
56	.24313860780E+79	-.18122458395E+79	-.17978158770E+79	.24300243364E+79
57	-.33288093736E+81	.25060019056E+81	.24865243769E+81	-.33269725674E+81
58	.46409174967E+83	-.35275719088E+83	-.35007941010E+83	.46383939265E+83
59	-.65865623302E+85	.50531964692E+85	.50157125601E+85	-.65830320718E+85
60	.95130275433E+87	-.73641489071E+87	-.73107411051E+87	.95080006765E+87
61	-.13978264043E+90	.10914896365E+90	.10837464339E+90	-.13970980316E+90
62	.20889890994E+92	-.16448871522E+92	-.16334672253E+92	.20879154920E+92
63	-.31742827946E+94	.25197342825E+94	.25026062350E+94	-.31726734556E+94
64	.49030189340E+96	-.39224913161E+96	-.38963737065E+96	.49005662669E+96
65	-.76961904301E+98	.62036586105E+98	.61631802287E+98	-.76923911493E+98
66	.12273591589+101	-.99656561860+100	-.99019085802+100	.12267611297+101
67	-.19881260886+103	.16256712706+103	.16154725411+103	-.19871697938+103

TABLE 6. COEFFICIENTS FOR ASYMPTOTIC EXPANSIONS OF CROSS PRODUCTS OF MODIFIED BESSEL FUNCTIONS FOR WALL RATIO B= 1.25

N	T(N)	U(N)	V(N)	W(N)
1	.44721359550E+00	.44721359550E+00	.44721359550E+00	.44721359550E+00
2	.11180339887E-01	.19006577809E+00	-.21242645786E+00	-.33541019662E-01
3	.45979147787E-01	.14674196102E-01	-.15512721594E-01	-.13626039238E+00
4	.15355498064E-01	.61474400100E-01	-.64933317753E-01	-.25522619649E-01
5	.66743026089E-01	.34145347605E-01	-.35426064273E-01	-.10953075459E+00
6	.66131893316E-01	.15318197601E+00	-.15877063377E+00	-.92212017775E-01
7	.31130374366E+00	.20836559768E+00	-.21416029871E+00	-.42887159988E+00
8	.59526478295E+00	.10224669607E+01	-.10498587005E+01	-.76172934756E+00
9	.30869065172E+01	.24356338180E+01	-.24893026205E+01	-.39075440330E+01
10	.92448393508E+01	.13250635270E+02	-.13530074403E+02	-.11241680172E+02
11	.53336006678E+02	.46637000575E+02	-.47486144477E+02	-.64244959245E+02
12	.22145817226E+03	.28317490275E+03	-.28809198775E+03	-.26034101580E+03
13	.14269283390E+04	.13304986693E+04	-.13511068223E+04	-.16638881803E+04
14	.76004475392E+04	.90297582923E+04	-.91629822680E+04	-.87228640337E+04
15	.54669927993E+05	.53048738260E+05	-.53762828617E+05	-.62311636009E+05
16	.35463766659E+06	.40158023114E+06	-.40673294447E+06	-.39978021349E+06
17	.28384978184E+07	.28223538679E+07	-.28558877133E+07	-.31811724170E+07
18	.21632309133E+08	.23729762120E+08	-.24000248835E+08	-.24049632217E+08
19	.19172503620E+09	.19346391815E+09	-.19548578242E+09	-.21211834663E+09
20	.16734884051E+10	.17961824027E+10	-.18158223546E+10	-.18394430614E+10
21	.16318049775E+11	.16641399252E+11	-.16751040079E+11	-.17893594769E+11
22	.16065995682E+12	.16851281118E+12	-.17206999170E+12	-.17400425977E+12
23	.16891303178E+13	.17908301627E+13	-.17235609663E+13	-.18801304123E+13
24	.19587263753E+14	.17674142913E+14	-.21433130875E+14	-.19120924374E+14
25	.16962854143E+15	.30393099359E+15	-.14222905919E+15	-.27970804093E+15

TABLE 6. COEFFICIENTS FOR ASYMPTOTIC EXPANSIONS OF CROSS PRODUCTS OF MODIFIED BESSEL FUNCTIONS FOR WALL RATIO B= 1.25 (CONT'D)

N	T(N)	U(N)	V(N)	W(N)
26	.48565261056E+16	-.12304133836E+16	-.65539102865E+16	-.49401411325E+15
27	-.83624098914E+17	.22671349320E+18	.15720842117E+18	-.15049685028E+18
28	.65534729140E+19	-.95239539994E+19	-.10258353208E+20	.56687032806E+19
29	-.33060530555E+21	.53838280422E+21	.51950826628E+21	-.34151526694E+21
30	.19181969030E+23	-.29463757183E+23	-.29241302830E+23	.18971803042E+23
31	-.11216552902E+25	.16995868826E+25	.16755292489E+25	-.11214611712E+25
32	.68233442572E+26	-.10101741087E+27	-.99794991917E+26	.68048498452E+26
33	-.42790484909E+28	.62132939612E+28	.61372175429E+28	-.42706592096E+28
34	.27691573890E+30	-.39451872432E+30	-.38984306338E+30	.27635328105E+30
35	-.18466394816E+32	.25843637620E+32	.25543566043E+32	-.18431159742E+32
36	.12680154487E+34	-.17447686273E+34	-.17249689560E+34	.12656809972E+34
37	-.89578592893E+35	.12129660824E+36	.11994958225E+36	-.89420267561E+35
38	.65055873316E+37	-.86760871984E+37	-.85817713322E+37	.64945144564E+37
39	-.48534518390E+39	.63800485973E+39	.63121221560E+39	-.48454895011E+39
40	.37170114184E+41	-.48197725061E+41	-.47694946921E+41	.37111258210E+41
41	-.29203047968E+43	.37378857650E+43	.36996659947E+43	-.29158366169E+43
42	.23522314050E+45	-.29739401075E+45	-.29441225484E+45	.23487498413E+45
43	-.19412842464E+47	.24258790801E+47	.24020205301E+47	-.19385017745E+47
44	.16406175726E+49	-.20275616348E+49	-.20079942407E+49	.16383381315E+49
45	-.14190511932E+51	.17353876726E+51	.17189484550E+51	-.14171382265E+51
46	.12555534854E+53	-.15201886237E+53	-.15060488396E+53	.12539097922E+53
47	-.11358022770E+55	.13622244371E+55	.13497797651E+55	-.11343570615E+55
48	.10500109267E+57	-.12480493556E+57	-.12368477769E+57	.10487113025E+57
49	-.99154031244E+58	.11685244439E+59	.11582178298E+59	-.99034560644E+58
50	.95601009163E+60	-.11175490805E+61	-.11078598594E+61	.95488792741E+60

TABLE 6. COEFFICIENTS FOR ASYMPTOTIC EXPANSIONS OF CROSS PRODUCTS OF MODIFIED BESSEL FUNCTIONS FOR WALL RATIO B= 1.25 (CONT'D)

N	T(N)	U(N)	V(N)	W(N)
51	-.94073620238E+62	.10912549392E+63	.10819524007E+63	-.93965972073E+62
52	.94439018001E+64	-.10875079591E+65	-.10783906725E+65	.94333598108E+64
53	-.96681866266E+66	.11056262702E+67	.10965082485E+67	-.96576519122E+66
54	.10089895688E+69	-.11462619676E+69	-.11369608954E+69	.10079157451E+69
55	-.10730515519E+71	.12114220355E+71	.12017482599E+71	-.10719354966E+71
56	.11625098277E+73	-.13046248613E+73	-.12943700145E+73	.11613275490E+73
57	-.12825375454E+75	.14312079977E+75	.14201321224E+75	-.12812614579E+75
58	.14404604327E+77	-.15988237293E+77	-.15866396174E+77	.14390575606E+77
59	-.16464767240E+79	.18181852669E+79	.18045383924E+79	-.16449064017E+79
60	.19147086244E+81	-.21041626731E+81	-.20886045339E+81	.19129194456E+81
61	-.22647319587E+83	.24773804099E+83	.24593322185E+83	-.22626576234E+83
62	.27238066019E+85	-.29665475289E+85	-.29452498487E+85	.27213601597E+85
63	-.33301465235E+87	.36118722472E+87	.35863141317E+87	-.33272122763E+87
64	.41377481778E+89	-.44700991425E+89	-.44389172325E+89	.41341701537E+89
65	-.52235771100E+91	.56219982112E+91	.55833314923E+91	-.52191424773E+91
66	.66983558548E+93	-.71835932592E+93	-.71348718606E+93	.66927708102E+93
67	-.87229009202E+95	.93231448845E+95	.92607799271E+95	-.87157552893E+95
68	.11533086622E+98	-.12287069087E+98	-.12205992771E+98	.11523801381E+98
69	-.15478340703+100	.16439855697+100	.16332832694+100	-.15466089423+100
70	.21081556180+102	-.22326118672+102	-.22182705062+102	.21065146296+102

TABLE 7. COEFFICIENTS FOR ASYMPTOTIC EXPANSIONS OF CROSS PRODUCTS OF MODIFIED BESSEL FUNCTIONS FOR WALL RATIO B= 1.30

N	T(N)	U(N)	V(N)	W(N)
1	.43852900965E+00	.43852900965E+00	.43852900965E+00	.43852900965E+00
2	.12649875278E-01	.18131487899E+00	-.20661462955E+00	-.37949625835E-01
3	.43808302046E-01	.16238061055E-01	-.17332761800E-01	-.12923550465E+00
4	.16815235005E-01	.57679384620E-01	-.61486276634E-01	-.27920131988E-01
5	.62768995528E-01	.36727344108E-01	-.38346091339E-01	-.10245666291E+00
6	.70501876515E-01	.14300914483E+00	-.14908365232E+00	-.98159680411E-01
7	.29232884520E+00	.21894704768E+00	-.22607875067E+00	-.40070233969E+00
8	.62115399412E+00	.95779002879E+00	-.98752107718E+00	-.79351910235E+00
9	.29129822956E+01	.25123769934E+01	-.25770365503E+01	-.36713663152E+01
10	.94874310601E+01	.12503615507E+02	-.12808369820E+02	-.11516440221E+02
11	.50700578570E+02	.47425347973E+02	-.48431531802E+02	-.60849391809E+02
12	.22440614698E+03	.26940320276E+03	-.27480404486E+03	-.26334644956E+03
13	.13664615530E+04	.13385651757E+04	-.13626774999E+04	-.15886047629E+04
14	.76292442163E+04	.86545176387E+04	-.88019005490E+04	-.87411199418E+04
15	.52686865237E+05	.52951205863E+05	-.53778910471E+05	-.59901165780E+05
16	.35353271138E+06	.38722972820E+06	-.39296415628E+06	-.39789328602E+06
17	.27493186765E+07	.28011650005E+07	-.28398042303E+07	-.30746809046E+07
18	.21457944442E+08	.22988904898E+08	-.23290230750E+08	-.23819989639E+08
19	.18640995129E+09	.19122063718E+09	-.19357899174E+09	-.20584448119E+09
20	.16538917592E+10	.17466955911E+10	-.17672569359E+10	-.18159252315E+10
21	.15919880839E+11	.16378992426E+11	-.16560973973E+11	-.17404940002E+11
22	.15793318574E+12	.16518060382E+12	-.16696685710E+12	-.17182488655E+12
23	.16706578605E+13	.17212877101E+13	-.17379172413E+13	-.18131935155E+13
24	.18332747779E+14	.19004615297E+14	-.19223594210E+14	-.19741587779E+14
25	.21038510347E+15	.21835397648E+15	-.21907607381E+15	-.22895308511E+15

TABLE 7. COEFFICIENTS FOR ASYMPTOTIC EXPANSIONS OF CROSS PRODUCTS OF MODIFIED BESSEL FUNCTIONS FOR WALL RATIO B= 1.30 (CONT'D)

N	T(N)	U(N)	V(N)	W(N)
26	.25774350433E+16	.25919594010E+16	-.26694530422E+16	-.26733236262E+16
27	.29889165655E+17	.33808521423E+17	-.31791122559E+17	-.35856575334E+17
28	.49189953019E+18	.37339797332E+18	-.47874077315E+18	-.36081019481E+18
29	.20371252285E+19	.80902984175E+19	-.34661496911E+19	-.94736054615E+19
30	.24127286597E+21	-.32002871008E+20	-.19045046981E+21	.79918593504E+20
31	-.66696892808E+22	.66711384278E+22	.42165970205E+22	-.89876644057E+22
32	.40305849599E+24	-.26199936511E+24	-.29108424529E+24	.36724905848E+24
33	-.19487544604E+26	.14897784660E+26	.14060838178E+26	-.19988725139E+26
34	.10479030140E+28	-.78528844061E+27	-.77871872093E+27	.10371154612E+28
35	-.56835885006E+29	.43983772343E+29	.43028508481E+29	-.56866223445E+29
36	.31996092632E+31	-.25126936077E+31	-.24704660426E+31	.31909959146E+31
37	-.18513029943E+33	.14805125174E+33	.14547923258E+33	-.18482318681E+33
38	.11029920264E+35	-.89587815729E+34	-.88120462943E+34	.11008801957E+35
39	-.67558432615E+36	.55692733021E+36	.54806619372E+36	-.67437960252E+36
40	.42518512014E+38	-.35533024065E+38	-.34987896195E+38	.42443214264E+38
41	-.27474129957E+40	.23254737440E+40	.22909363855E+40	-.27426743960E+40
42	.18215089086E+42	-.15601224789E+42	-.15376833487E+42	.18184292545E+42
43	-.12382718297E+44	.10723193632E+44	.10573624873E+44	-.12362227077E+44
44	.86260369328E+45	-.75468343723E+45	-.74446524813E+45	.86120524496E+45
45	-.61540639038E+47	.54356289505E+47	.53641149425E+47	-.61442882470E+47
46	.44939156705E+49	-.40045875527E+49	-.39533449182E+49	.44869185517E+49
47	-.33571273111E+51	.30163169979E+51	.29787450642E+51	-.33520022653E+51
48	.25642994635E+53	-.23216904864E+53	-.22935156710E+53	.25604600527E+53
49	-.20017756119E+55	.18253453476E+55	.18037475553E+55	-.19988352530E+55
50	.15962631554E+57	-.14652514011E+57	-.14483352145E+57	.15939622492E+57

TABLE 7. COEFFICIENTS FOR ASYMPTOTIC EXPANSIONS OF CROSS PRODUCTS OF MODIFIED BESSEL FUNCTIONS FOR WALL RATIO B= 1.30 (CONT'D)

N	T(N)	U(N)	V(N)	W(N)
51	-.12996943554E+59	.12003998942E+59	.11868684754E+59	-.12978554399E+59
52	.10800374264E+61	-.10032604077E+61	-.99221089798E+60	.10785370540E+61
53	-.91562553696E+62	.85508454206E+62	.84587749832E+62	-.91437634702E+62
54	.79160088714E+64	-.74293698940E+64	-.73511169216E+64	.79053998558E+64
55	-.69765124095E+66	.65779135298E+66	.65100998020E+66	-.69673254717E+66
56	.62654884600E+68	-.59329243326E+68	-.58730267045E+68	.62573797102E+68
57	-.57319535175E+70	.54494211070E+70	.53955169840E+70	-.57246611442E+70
58	.53399092944E+72	-.50955871480E+72	-.50461782099E+72	.53332294563E+72
59	-.50641465258E+74	.48491598133E+74	.48030475497E+74	-.50579163451E+74
60	.48874523241E+76	-.46950230133E+76	-.46512187820E+76	.48815376096E+76
61	-.47987730837E+78	.46236263710E+78	.45812844524E+78	-.47930592050E+78
62	.47920504139E+80	-.46299934579E+80	-.45883592233E+80	.47864352484E+80
63	-.48655554528E+82	.47131729307E+82	.46715405441E+82	-.48599436410E+82
64	.50216203095E+84	-.48760495648E+84	-.48337250559E+84	.50159182543E+84
65	-.52667189230E+86	.51254792587E+86	.50817456207E+86	-.52608300755E+86
66	.56118930637E+88	-.54727513951E+88	-.54268327306E+88	.56057130945E+88
67	-.60735597981E+90	.59344198402E+90	.58854415828E+90	-.60669712526E+90
68	.66747807503E+92	-.65335860708E+92	-.64805279670E+92	.66676467421E+92
69	-.74471274249E+94	.73017706049E+94	.72434084179E+94	-.74392838277E+94
70	.84333486567E+96	-.82815797941E+96	-.82164104955E+96	.84245940925E+96
71	-.96911466777E+98	.95304747389E+98	.94566181283E+98	-.96812293591E+98
72	.11298512901+101	-.11126093052+101	-.11041160036+101	.11287113021+101
73	-.13361286451+103	.13173785490+103	.13074699263+103	-.13347992289+103

TABLE 8. COEFFICIENTS FOR ASYMPTOTIC EXPANSIONS OF CROSS PRODUCTS OF MODIFIED BESSEL FUNCTIONS FOR WALL RATIO B= 1.40

N	T(N)	U(N)	V(N)	W(N)
1	.42257712736E+00	.42257712736E+00	.42257712736E+00	.42257712736E+00
2	.15092040263E-01	.16601244289E+00	-.19619652342E+00	-.45276120789E-01
3	.40155607128E-01	.18595549610E-01	-.20212553924E-01	-.11723281276E+00
4	.18913175457E-01	.51542012505E-01	-.55873274060E-01	-.31329458582E-01
5	.56622221371E-01	.40019052077E-01	-.42274917470E-01	-.91362548427E-01
6	.75754215771E-01	.12775191309E+00	-.13456011484E+00	-.10512107146E+00
7	.26499840106E+00	.22959420505E+00	-.23909276799E+00	-.35959948596E+00
8	.64508019629E+00	.86647377529E+00	-.89995760054E+00	-.82104246560E+00
9	.26752832918E+01	.25614590365E+01	-.26447122359E+01	-.33445034331E+01
10	.96145871986E+01	.11489923385E+02	-.11837491767E+02	-.11627219568E+02
11	.47202136579E+02	.47401011622E+02	-.48665310347E+02	-.56286723903E+02
12	.22356801941E+03	.25100403880E+03	-.25724505582E+03	-.26141650456E+03
13	.12864868993E+04	.13198439616E+04	-.13496294464E+04	-.14879003523E+04
14	.75133299826E+04	.81489609442E+04	-.83211163628E+04	-.85790499419E+04
15	.50011484313E+05	.51741905241E+05	-.52752546141E+05	-.56617176330E+05
16	.34549993357E+06	.36734208449E+06	-.37409331989E+06	-.38762360022E+06
17	.26244319294E+07	.27212315727E+07	-.27680475063E+07	-.29244191945E+07
18	.20865800713E+08	.21918756039E+08	-.22275404645E+08	-.23095086595E+08
19	.17860332283E+09	.18507586824E+09	-.18791952615E+09	-.19660454775E+09
20	.16030765226E+10	.16709558391E+10	-.16953555734E+10	-.17554408374E+10
21	.15290020488E+11	.15815862422E+11	-.16035331144E+11	-.16668477122E+11
22	.15274756807E+12	.15837135597E+12	-.16046956161E+12	-.16581919791E+12
23	.16076750449E+13	.16593907919E+13	-.16803693491E+13	-.17388096741E+13
24	.17681496812E+14	.18261404878E+14	-.18483159331E+14	-.19057065182E+14
25	.20363881886E+15	.20974405537E+15	-.21216973782E+15	-.21881555368E+15

TABLE 8. COEFFICIENTS FOR ASYMPTOTIC EXPANSIONS OF CROSS PRODUCTS OF MODIFIED BESSEL FUNCTIONS FOR WALL RATIO B= 1.40 (CONT'D)

N	T(N)	U(N)	V(N)	W(N)
26	.24443929168E+16	.25167677646E+16	-.25452537147E+16	-.26184152617E+16
27	.30578659100E+17	.31441344329E+17	-.31767771246E+17	-.32685583906E+17
28	.39792440904E+18	.40841119203E+18	-.41301864278E+18	-.42374598216E+18
29	.53671599394E+19	.55192641724E+19	-.55617551262E+19	-.57207622520E+19
30	.75482138363E+20	.76939020426E+20	-.78126560491E+20	-.79640683607E+20
31	.10838255250E+22	.11264813924E+22	-.11208981049E+22	-.11644468273E+22
32	.16687972533E+23	.16467201448E+23	-.17220438415E+23	-.17018877827E+23
33	.24235711873E+24	.27165355532E+24	-.25061761199E+24	-.27982009071E+24
34	.45362163079E+25	.36853814755E+25	-.46536357245E+25	-.38172354563E+25
35	.47781831874E+26	.88015078803E+26	-.50090182498E+26	-.89897198177E+26
36	.19638652589E+28	.33896287568E+27	-.19845956946E+28	-.37988570456E+27
37	-.15978414995E+29	.56542794944E+29	.14899892154E+29	-.56804127943E+29
38	.19779508343E+31	-.12598331378E+31	-.19656477076E+31	.12356715308E+31
39	-.68164551250E+32	.82017710749E+32	.67006359542E+32	-.81547751069E+32
40	.36801723765E+34	-.34442052485E+34	-.36388130606E+34	.34100580137E+34
41	-.17074612026E+36	.17668636925E+36	.16859376254E+36	-.17525726499E+36
42	.87090823134E+37	-.86575011599E+37	-.86097972634E+37	.85821561843E+37
43	-.44274848575E+39	.44725725767E+39	.43776181319E+39	-.44351611826E+39
44	.23316465700E+41	-.23402864093E+41	-.23065596992E+41	.23206506605E+41
45	-.12515497207E+43	.12594005284E+43	.12384835655E+43	-.12489931790E+43
46	.68865045759E+44	-.69224017409E+44	-.68170455377E+44	.68657161778E+44
47	-.38725919454E+46	.38942522125E+46	.38347191270E+46	-.38627598382E+46
48	.22267806958E+48	-.22387788737E+48	-.22056589532E+48	.22208901745E+48
49	-.13081154971E+50	.13151916725E+50	.12960626362E+50	-.13048176853E+50
50	.78483876905E+51	-.78902481448E+51	-.77780764349E+51	.78288249029E+51

TABLE 8. COEFFICIENTS FOR ASYMPTOTIC EXPANSIONS OF CROSS PRODUCTS OF MODIFIED BESSEL FUNCTIONS FOR WALL RATIO B= 1.40 (CONT'D)

N	T(N)	U(N)	V(N)	W(N)
51	-.48068860320E+53	.48323206594E+53	.47649699331E+53	-.47952094289E+53
52	.30041819580E+55	-.30198973626E+55	-.29786593637E+55	.29970235566E+55
53	-.19150902636E+57	.19250002160E+57	.18992242450E+57	-.19106231840E+57
54	.12447652875E+59	-.12511309869E+59	-.12347002558E+59	.12419189489E+59
55	-.82463046597E+60	.82879722604E+60	.81811740989E+60	-.82278198222E+60
56	.55660932132E+62	-.55938705019E+62	-.55231197431E+62	.55538554200E+62
57	-.38265867464E+64	.38454424024E+64	.37976870456E+64	-.38183322569E+64
58	.26785451968E+66	-.26915742783E+66	-.26587435543E+66	.26728740675E+66
59	-.19084205921E+68	.19175826840E+68	.18946018626E+68	-.19044534320E+68
60	.13835763645E+70	-.13901312415E+70	-.13737578893E+70	.13807515580E+70
61	-.10203681566E+72	.10251380725E+72	.10132677257E+72	-.10183214320E+72
62	.76526267487E+73	-.76879223997E+73	-.76003809746E+73	.76375411191E+73
63	-.58350330455E+75	.58615844352E+75	.57959300870E+75	-.58237253247E+75
64	.45220824393E+77	-.45423827385E+77	-.44923225656E+77	.45134650948E+77
65	-.35610900238E+79	.35768611562E+79	.35380654709E+79	-.35544151760E+79
66	.28488348351E+81	-.28612819048E+81	-.28307310249E+81	.28435811323E+81
67	-.23146500978E+83	.23246274694E+83	.23001873243E+83	-.23104492586E+83
68	.19095645579E+85	-.19176856285E+85	-.18978284501E+85	.19061530478E+85
69	-.15992432180E+87	.16059538617E+87	.15895720405E+87	-.15964300593E+87
70	.13593430763E+89	-.13649713696E+89	-.13512518792E+89	.13569881420E+89
71	-.11724223081E+91	.11772125894E+91	.11655512572E+91	-.11704215099E+91
72	.10258603570E+93	-.10299968217E+93	-.10199391190E+93	.10241354189E+93
73	-.91044337629E+94	.91406660695E+94	.90526628373E+94	-.90893469311E+94
74	.81939247965E+96	-.82261115651E+96	-.81480101322E+96	.81805407323E+96
75	-.74768995492E+98	.75058924638E+98	.74356024292E+98	-.74648587025E+98

TABLE 9. COEFFICIENTS FOR ASYMPTOTIC EXPANSIONS OF CROSS PRODUCTS OF MODIFIED BESSEL FUNCTIONS FOR WALL RATIO B= 1.50

N	T(N)	U(N)	V(N)	W(N)
1	.40824829046E+00	.40824829046E+00	.40824829046E+00	.40824829046E+00
2	.17010345436E-01	.15309310892E+00	-.18711379980E+00	-.51031036308E-01
3	.37210130641E-01	.20199785205E-01	-.22326078385E-01	-.10737780556E+00
4	.20244082980E-01	.46822747724E-01	-.51518311828E-01	-.33444819803E-01
5	.52127867664E-01	.41688358789E-01	-.44507727567E-01	-.83120621070E-01
6	.78133372270E-01	.11694699006E+00	-.12428553936E+00	-.10802687179E+00
7	.24640685773E+00	.23222378315E+00	-.24366798776E+00	-.33123655529E+00
8	.64885545404E+00	.80513404088E+00	-.84157466982E+00	-.82262653828E+00
9	.25193877884E+01	.25410791722E+01	-.26389187682E+01	-.31273961896E+01
10	.95161070735E+01	.10819870492E+02	-.11202737895E+02	-.11464408013E+02
11	.44902056127E+02	.46460005985E+02	-.47922089822E+02	-.53255753226E+02
12	.21908903711E+03	.23861044854E+03	-.24555146315E+03	-.25527172846E+03
13	.12320429788E+04	.12843949230E+04	-.13184975501E+04	-.14188479272E+04
14	.73193199072E+04	.77934141398E+04	-.79860974684E+04	-.83304662856E+04
15	.48091035854E+05	.50146322997E+05	-.51296512132E+05	-.54250991534E+05
16	.33542253678E+06	.35259664066E+06	-.36018184523E+06	-.37521303712E+06
17	.25295979656E+07	.26313870043E+07	-.26844790648E+07	-.28102461635E+07
18	.20217497258E+08	.21084137017E+08	-.21485708520E+08	-.22318014691E+08
19	.17237414449E+09	.17875349532E+09	-.18197177991E+09	-.18924586020E+09
20	.15515506102E+10	.16093519683E+10	-.16368560982E+10	-.16949129847E+10
21	.14767085008E+11	.15266578927E+11	-.15514686316E+11	-.16060349346E+11
22	.14774577126E+12	.15264627193E+12	-.15501226295E+12	-.16003605766E+12
23	.15532601176E+13	.16013290787E+13	-.16250434077E+13	-.16763460726E+13
24	.17096311853E+14	.17609369651E+14	-.17859084171E+14	-.18389456791E+14
25	.19678655595E+15	.20238083978E+15	-.20513350248E+15	-.21102608766E+15

TABLE 9. COEFFICIENTS FOR ASYMPTOTIC EXPANSIONS OF CROSS PRODUCTS OF MODIFIED BESSEL FUNCTIONS FOR WALL RATIO B= 1.50 (CONT'D)

N	T(N)	U(N)	V(N)	W(N)
26	.23628632781E+16	.24277471220E+16	-.24594748606E+16	-.25267188042E+16
27	.29556849097E+17	.30332567288E+17	-.30714018726E+17	-.31524687932E+17
28	.38445243238E+18	.39419364603E+18	-.39897067538E+18	-.40906492645E+18
29	.51931474084E+19	.53196138994E+19	-.53818209031E+19	-.55133142233E+19
30	.72740248758E+20	.74452139633E+20	-.75293191141E+20	-.77064887184E+20
31	.10552579852E+22	.10792168278E+22	-.10910111807E+22	-.11158334535E+22
32	.15835847266E+23	.16183897504E+23	-.16355075743E+23	-.16714683485E+23
33	.24556212494E+24	.25078193248E+24	-.25335525498E+24	-.25874858082E+24
34	.39305045540E+25	.40115774666E+25	-.40514367673E+25	-.41350542923E+25
35	.64878977650E+26	.66174268449E+26	-.66815225402E+26	-.68150709009E+26
36	.11032882445E+28	.11247623977E+28	-.11352651847E+28	-.11573743884E+28
37	.19315454437E+29	.19677766997E+29	-.19859440665E+29	-.20232373249E+29
38	.34773487879E+30	.35418528377E+30	-.35726171345E+30	-.36389031098E+30
39	.64369657531E+31	.65492229823E+31	-.66085619175E+31	-.67240067857E+31
40	.12226281454E+33	.12448931230E+33	-.12543947915E+33	-.12772127422E+33
41	.23876596674E+34	.24241895915E+34	-.24480458168E+34	-.24856765609E+34
42	.47649561500E+35	.48594022131E+35	-.48828200734E+35	-.49790649605E+35
43	.98180660289E+36	.99120102759E+36	-.10053694510E+37	-.10152319478E+37
44	.20422990506E+38	.21014823725E+38	-.20907429564E+38	-.21503844192E+38
45	.44778437093E+39	.44305302178E+39	-.45790682948E+39	-.45347035590E+39
46	.94751028188E+40	.10128641661E+41	-.96955235142E+40	-.10346623599E+41
47	.23025947824E+42	.21065225186E+42	-.23498218026E+42	-.21570238135E+42
48	.45135557655E+43	.56358810994E+43	-.46260678955E+43	-.57385075846E+43
49	.14291834642E+45	.95245559863E+44	-.14520564103E+45	-.98096534860E+44
50	.16624730019E+46	.40697351744E+46	-.17310669129E+46	-.41143504727E+46

TABLE 9. COEFFICIENTS FOR ASYMPTOTIC EXPANSIONS OF CROSS PRODUCTS OF MODIFIED BESSEL FUNCTIONS FOR WALL RATIO B= 1.50 (CONT'D)

N	T(N)	U(N)	V(N)	W(N)
51	.12739167675E+48	.12306863841E+47	-.12830065179E+48	-.14425547289E+47
52	-.11479375861E+49	.46758931830E+49	.10883947376E+49	-.46775875879E+49
53	.19174149310E+51	-.10349719997E+51	-.19132019994E+51	.10099192481E+51
54	-.65078792969E+52	.88879969461E+52	.64191484582E+52	-.88283673938E+52
55	.43673407694E+54	-.37802332824E+54	-.43377841640E+54	.37324347136E+54
56	-.21106044880E+56	.22918410301E+56	.20901350498E+56	-.22713766745E+56
57	.12290549618E+58	-.11912873736E+58	-.12191116315E+58	.11791147946E+58
58	-.66920082679E+59	.68632699005E+59	.66333904084E+59	-.68003105042E+59
59	.38689558273E+61	-.38564038485E+61	-.38368141315E+61	.38204612465E+61
60	-.22282527045E+63	.22523535400E+63	.22095355429E+63	-.22322714896E+63
61	.13193435726E+65	-.13239988873E+65	-.13085036037E+65	.13123375820E+65
62	-.79021662039E+66	.79571034344E+66	.78376682803E+66	-.78888645540E+66
63	.48246136281E+68	-.48489388157E+68	-.47858260842E+68	.48080939968E+68
64	-.29899500759E+70	.30073336638E+70	.29661869634E+70	-.29825274586E+70
65	.18840875360E+72	-.18939936646E+72	-.18693182088E+72	.18786536586E+72
66	-.12056824211E+74	.12121650262E+74	.11963529318E+74	-.12025279233E+74
67	.78373549144E+75	-.78778559000E+75	-.77775299394E+75	.78163086611E+75
68	-.51724962510E+77	.51990368220E+77	.51335405518E+77	-.51591094515E+77
69	.34656070502E+79	-.34829898004E+79	-.34398595858E+79	.34566800927E+79
70	-.23565857398E+81	.23682426269E+81	.23393136198E+81	-.23506401228E+81
71	.16260414126E+83	-.16339443020E+83	-.16142844124E+83	.16219889846E+83
72	-.11382206658E+85	.11436702218E+85	.11301013421E+85	-.11354296727E+85
73	.80813258820E+86	-.81194154496E+86	-.80244499018E+86	.80617833732E+86
74	-.58185162651E+88	.58455388530E+88	.57781100821E+88	-.58046519415E+88
75	.42474908847E+90	-.42669295395E+90	-.42183839262E+90	.42375108063E+90

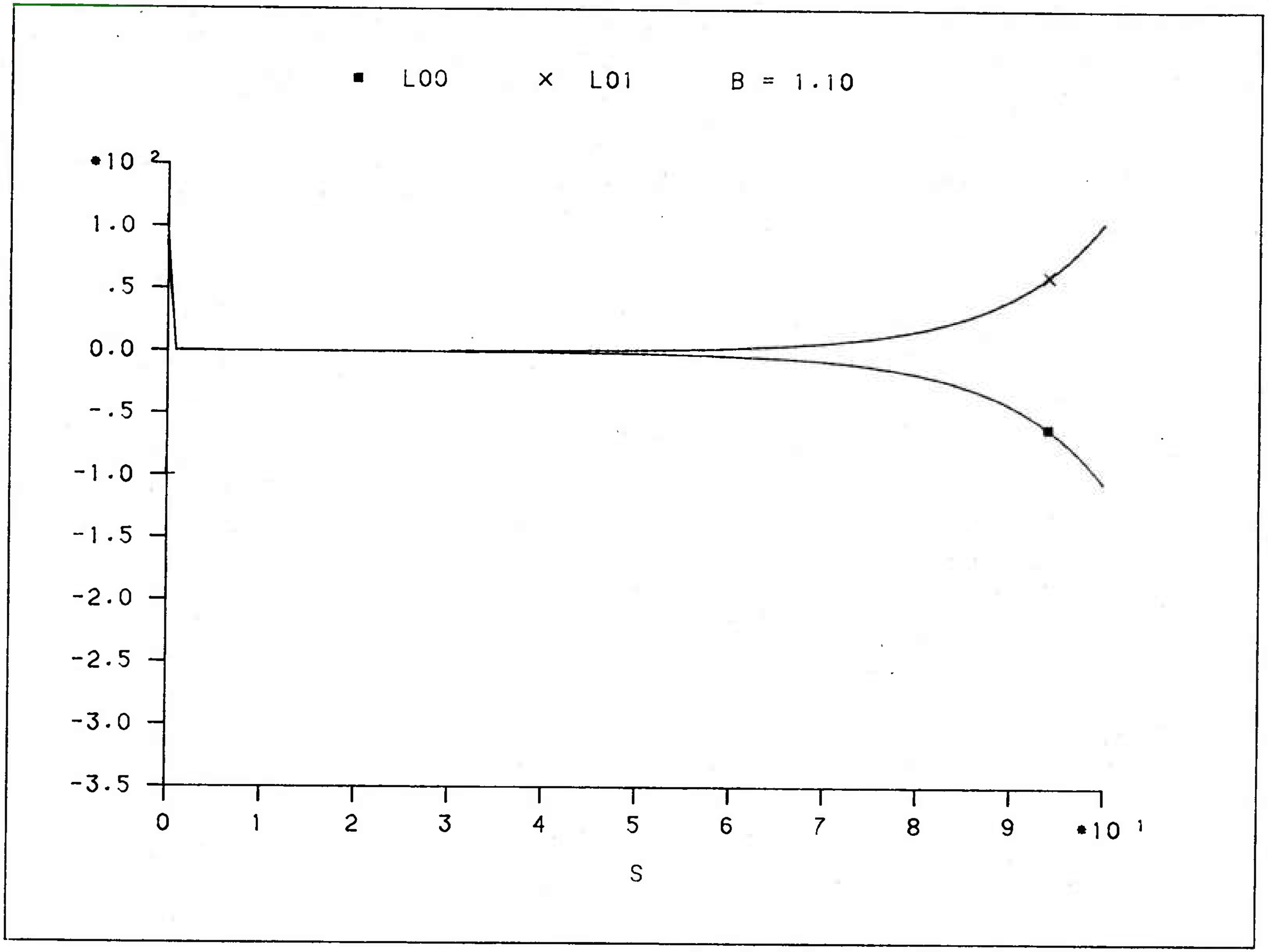

Figure 13. Graph of $L_{0,0}(s)$ and $L_{0,1}(s)$ for Wall Ratio = 1.1

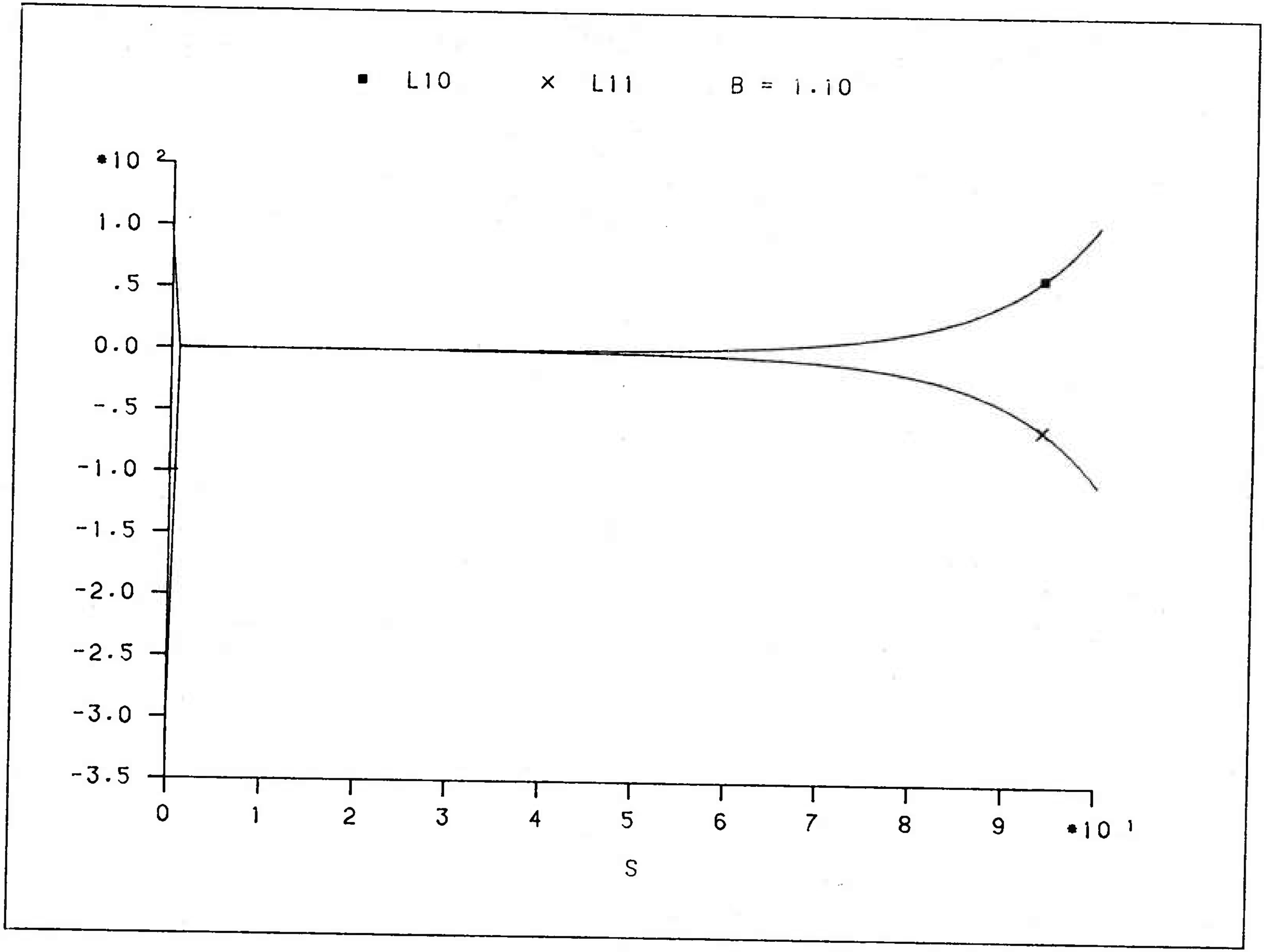

Figure 14. Graph of $L_{1,0}(s)$ and $L_{1,1}(s)$ for Wall Ratio = 1.1

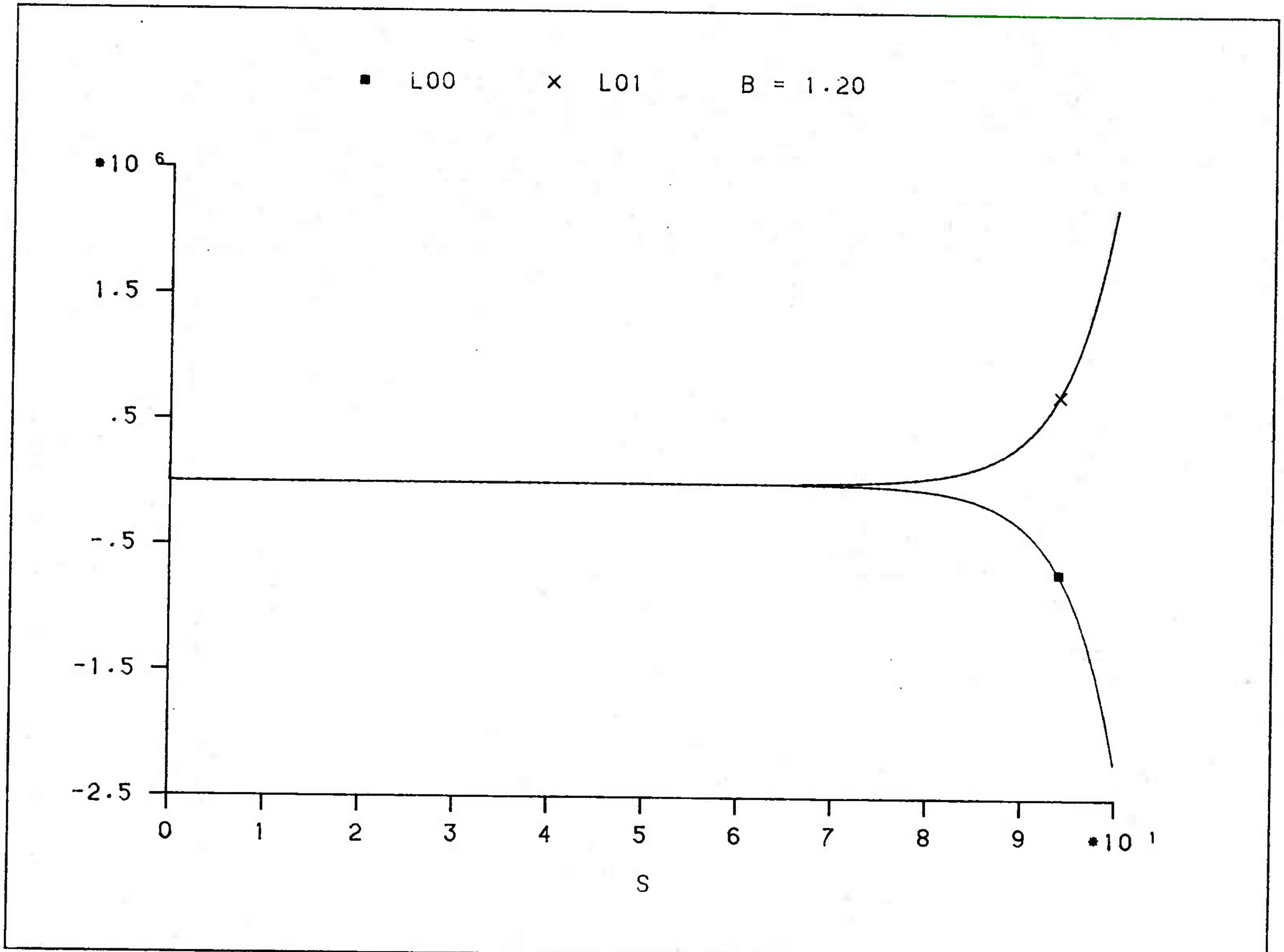

Figure 15. Graph of $L_{0,0}(s)$ and $L_{0,1}(s)$ for Wall Ratio = 1.2

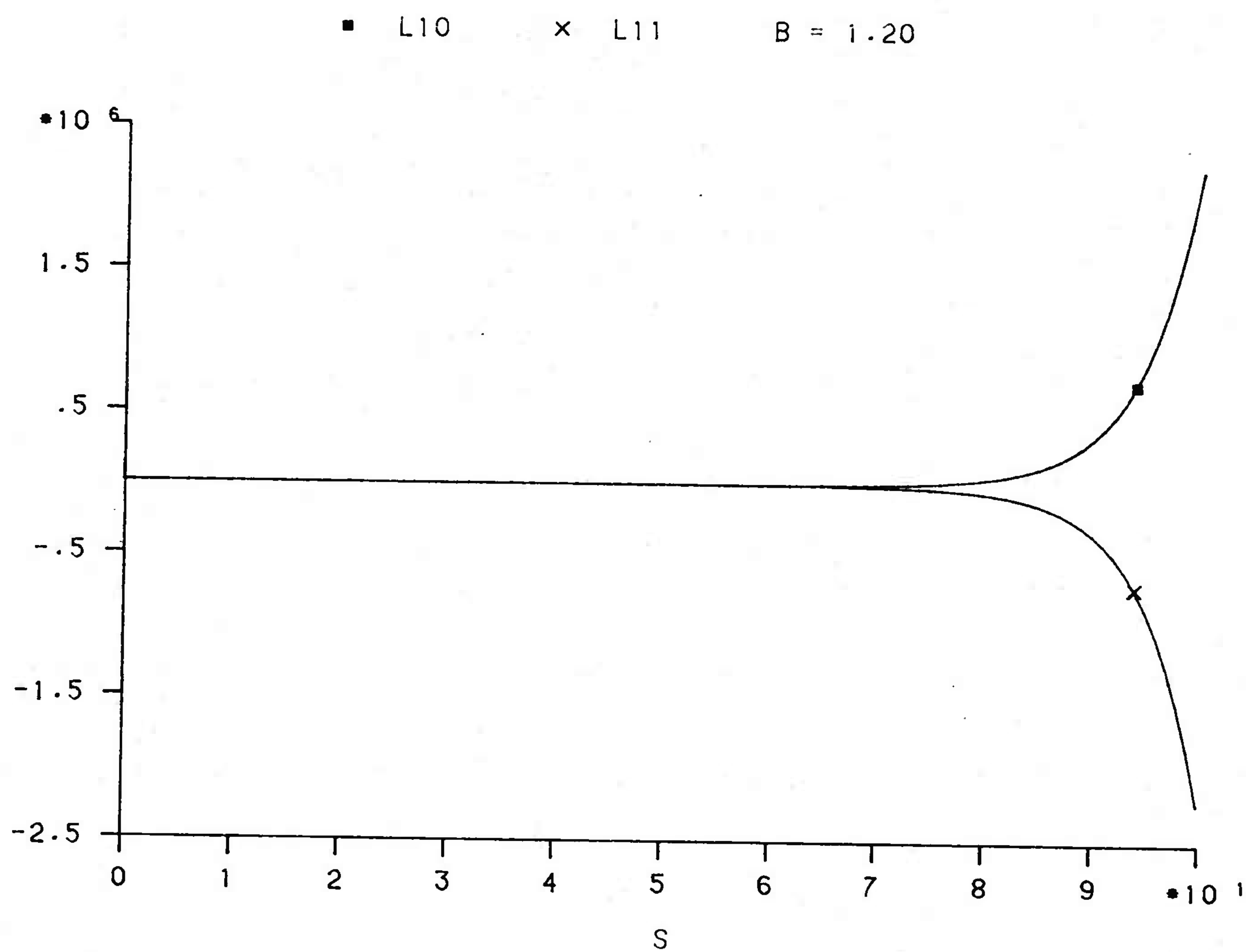

Figure 16. Graph of $L_{1,0}(s)$ and $L_{1,1}(s)$ for Wall Ratio = 1.2

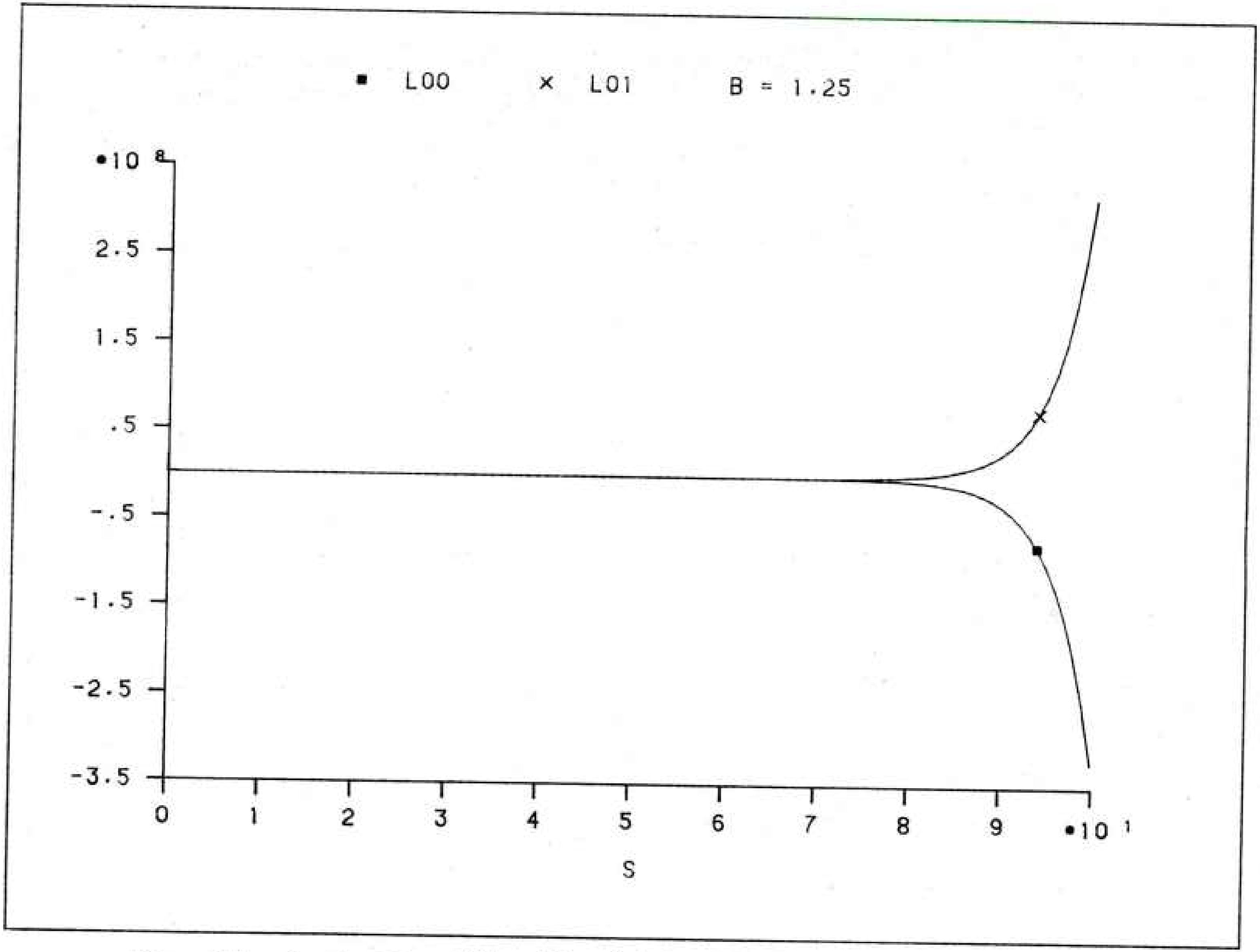

Figure 17. Graph of $L_{0,0}(s)$ and $L_{0,1}(s)$ for Wall Ratio = 1.25

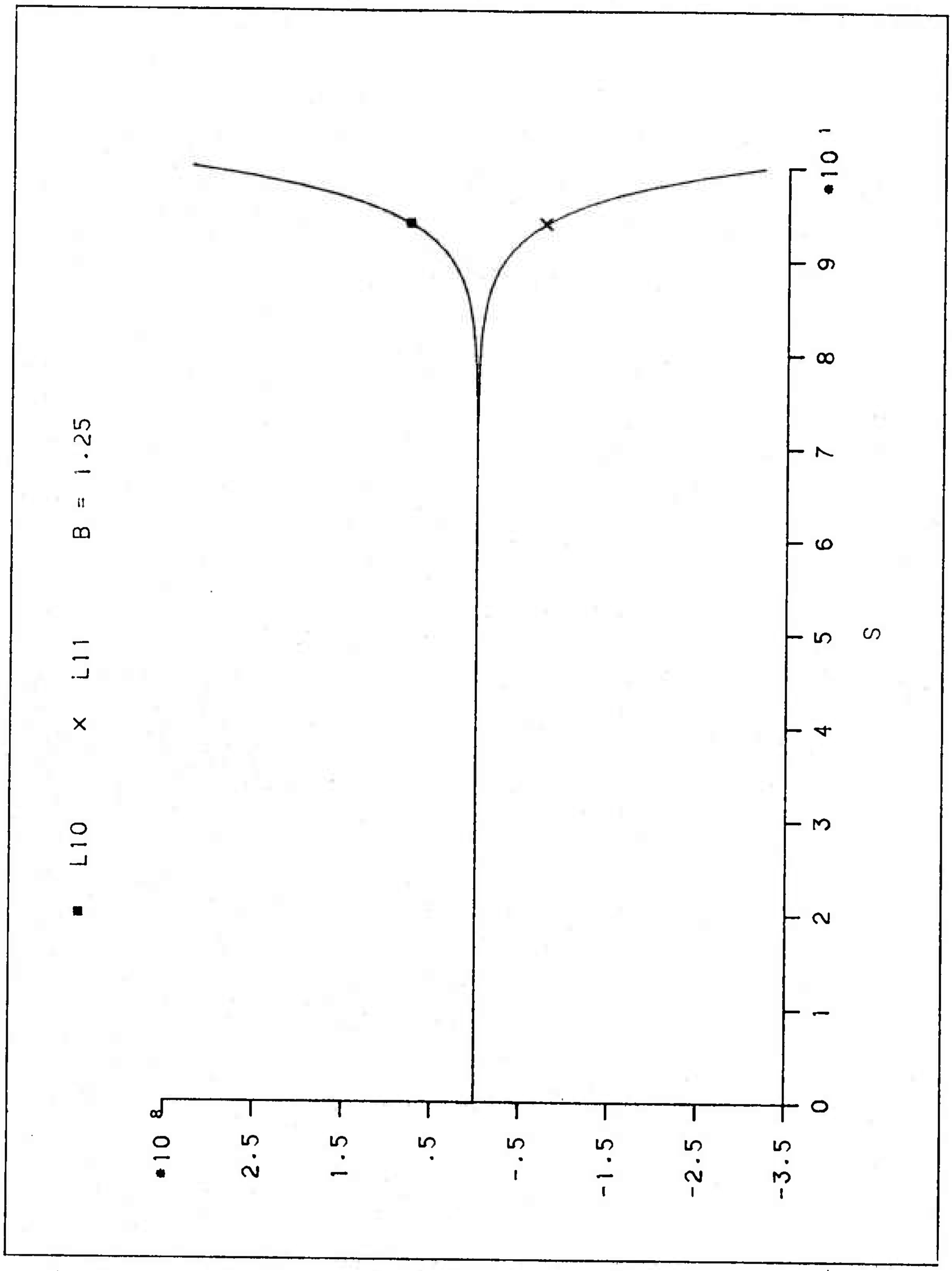

Figure 18. Graph of $L_{1,0}(s)$ and $L_{1,1}(s)$ for Wall Ratio = 1.25

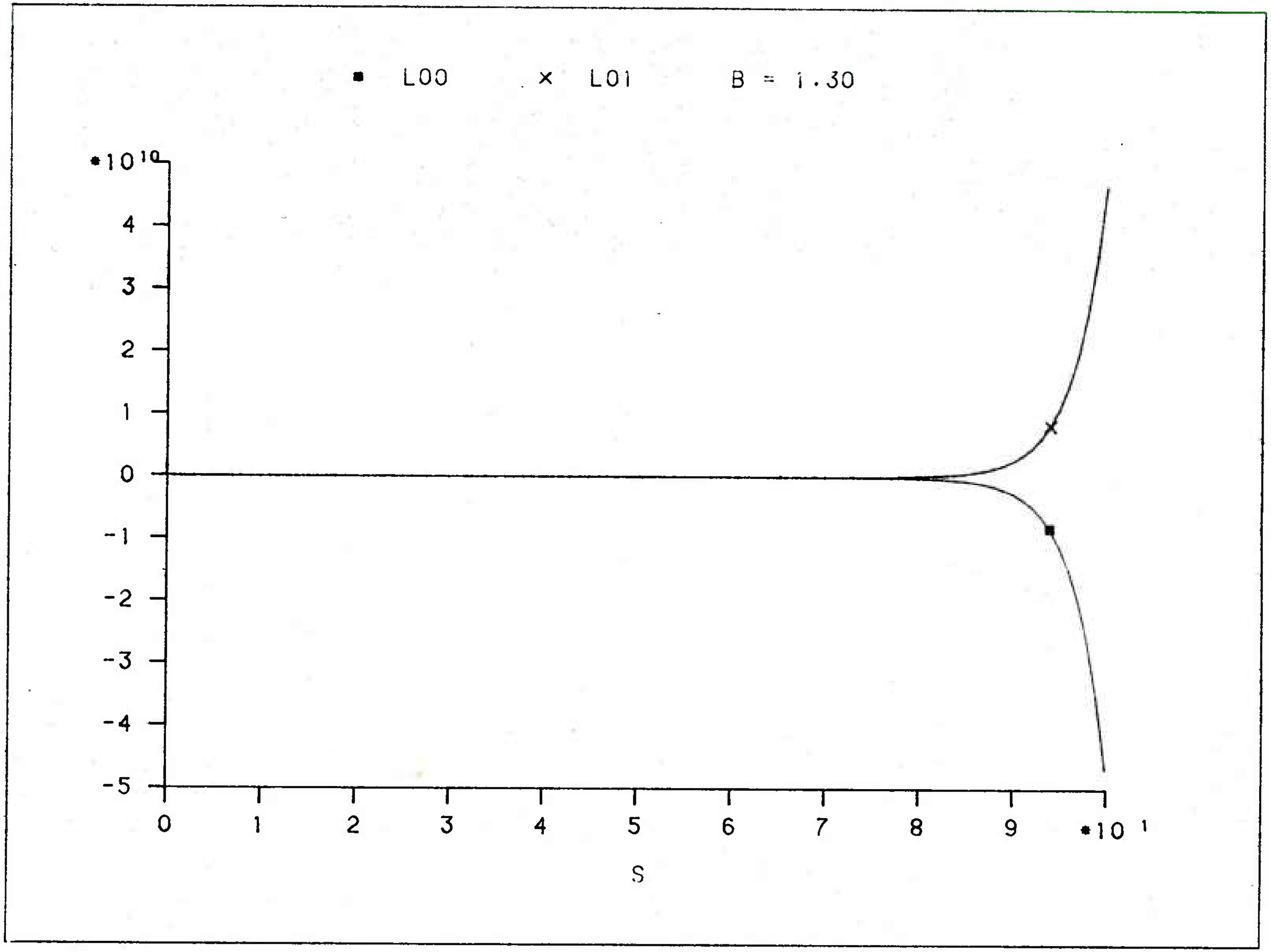

Figure 19. Graph of $L_{0,0}(s)$ and $L_{0,1}(s)$ for Wall Ratio = 1.3

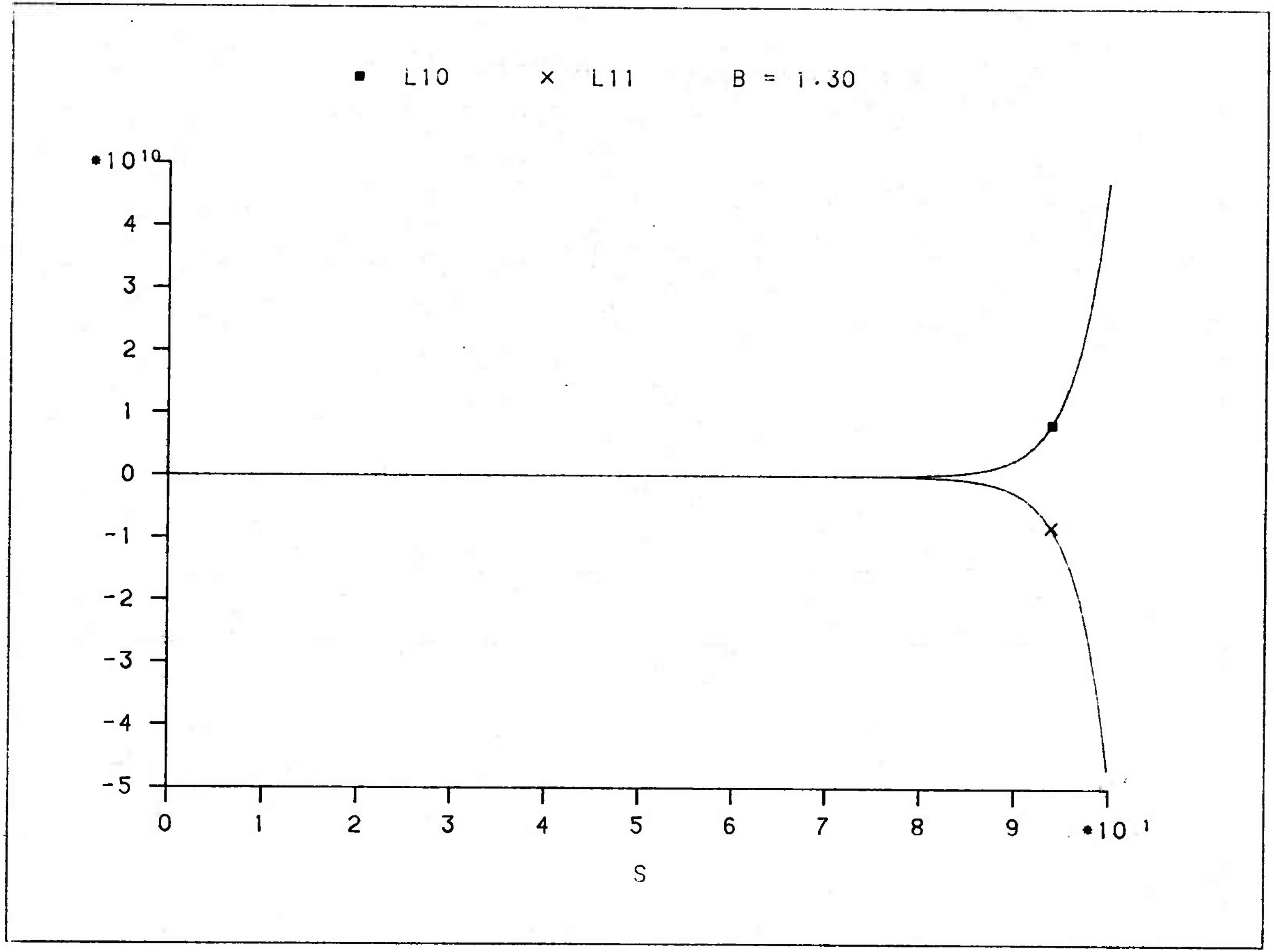

Figure 20. Graph of $L_{1,0}(s)$ and $L_{1,1}(s)$ for Wall Ratio = 1.3

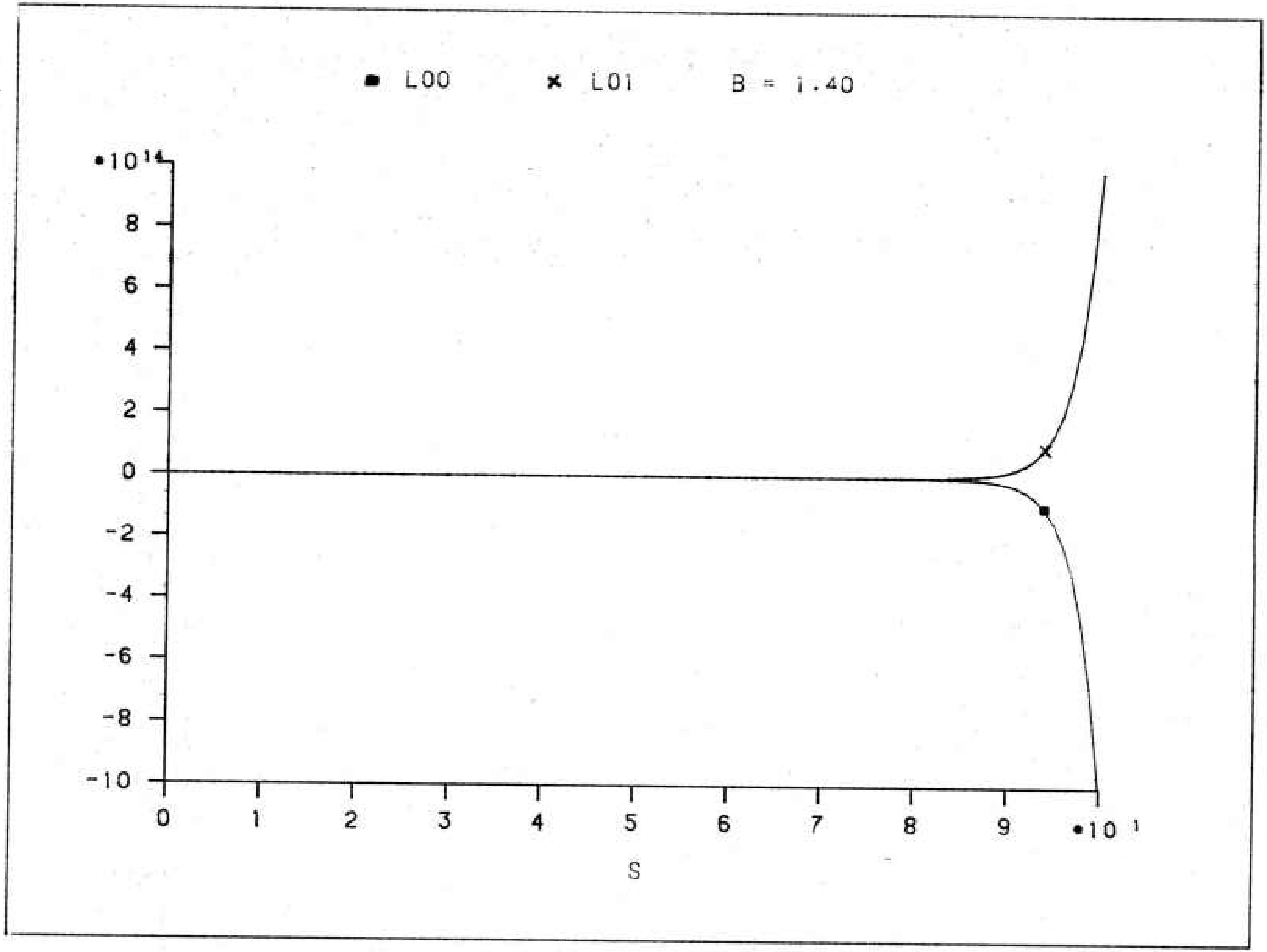

Figure 21. Graph of $L_{0,0}(s)$ and $L_{0,1}(s)$ for Wall Ratio = 1.4

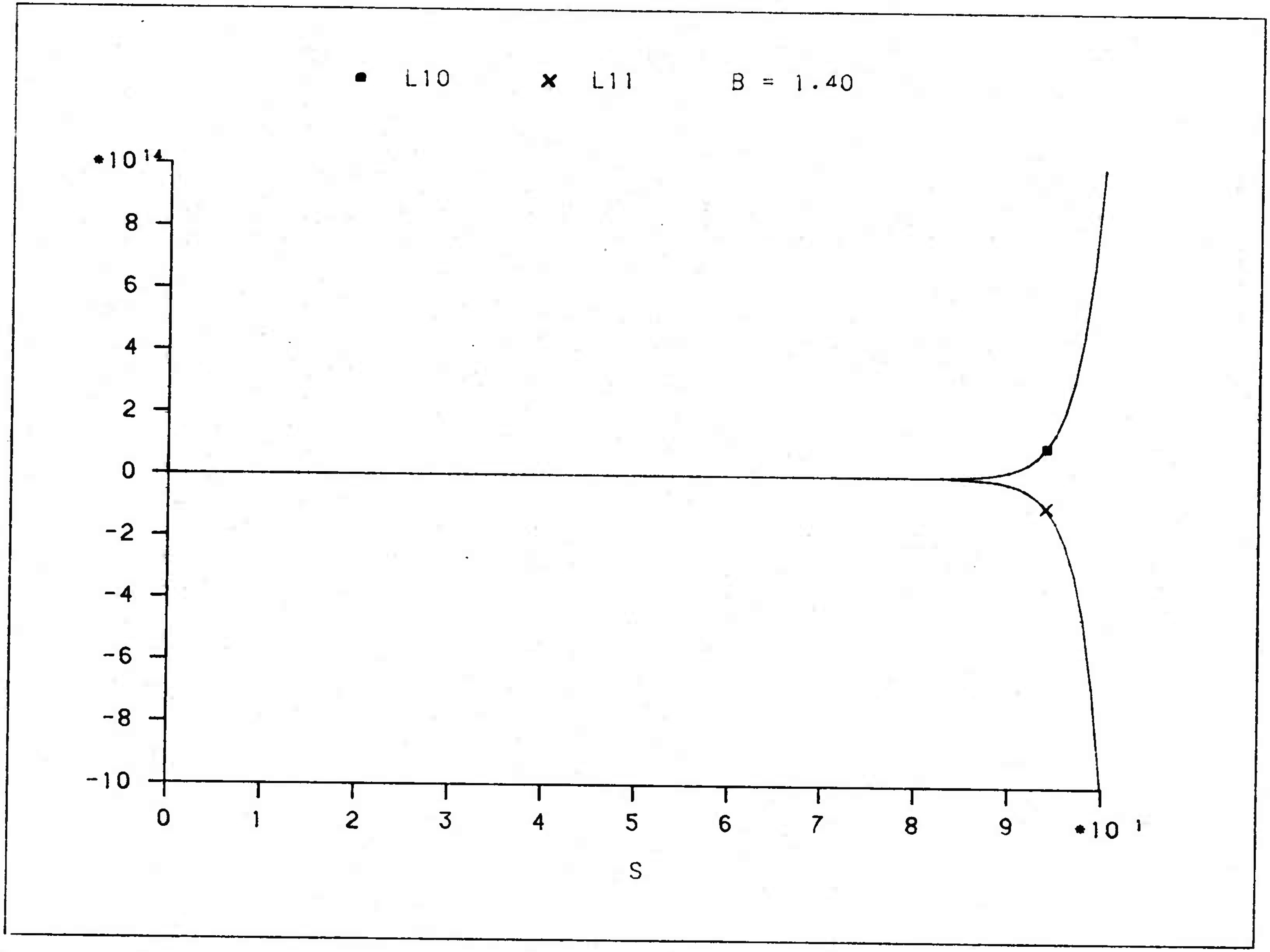

Figure 22. Graph of $L_{1,0}(s)$ and $L_{1,1}(s)$ for Wall Ratio = 1.4

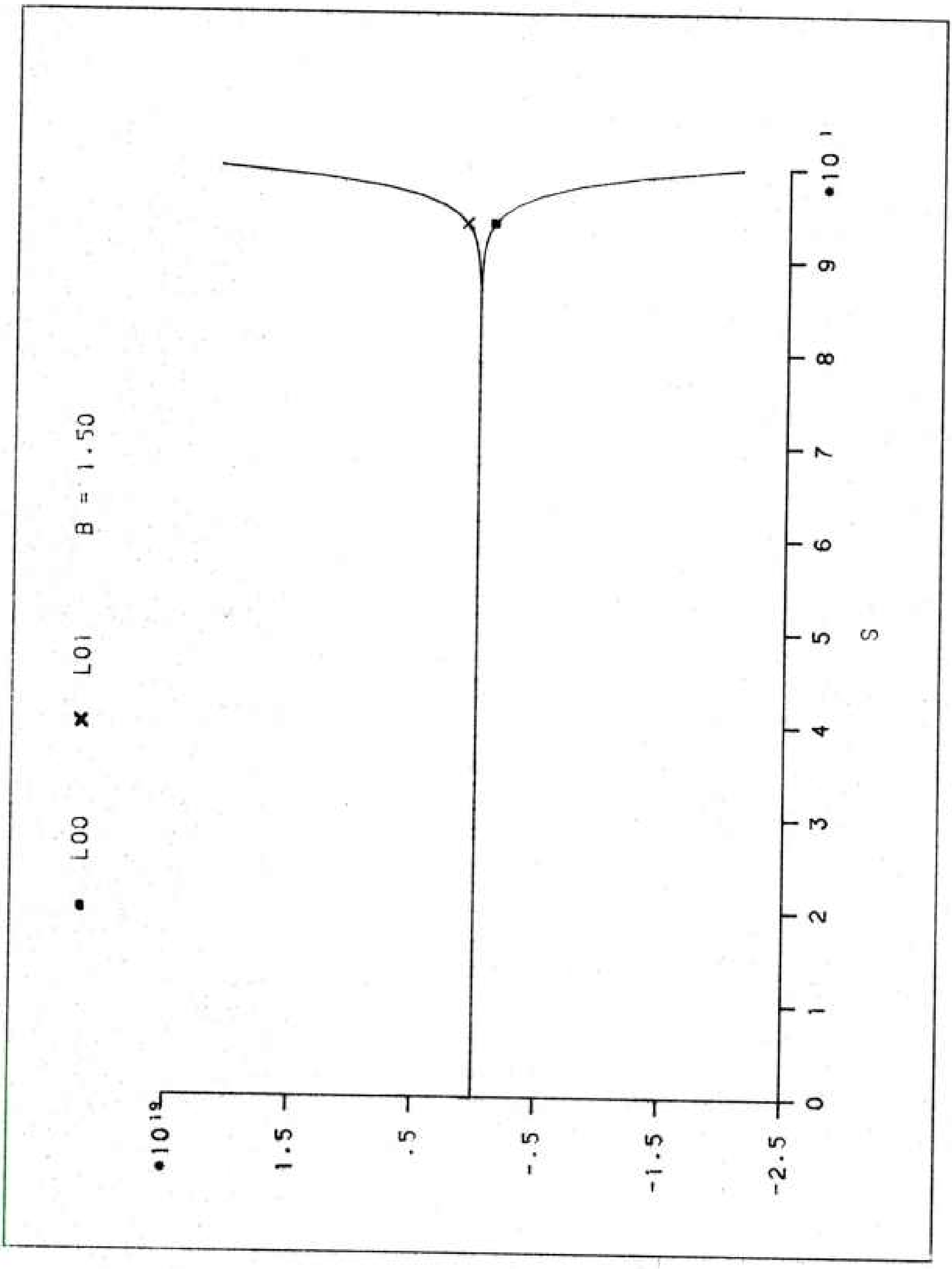

Figure 23. Graph of $L_{0,0}(s)$ and $L_{0,1}(s)$ for Wall Ratio = 1.5

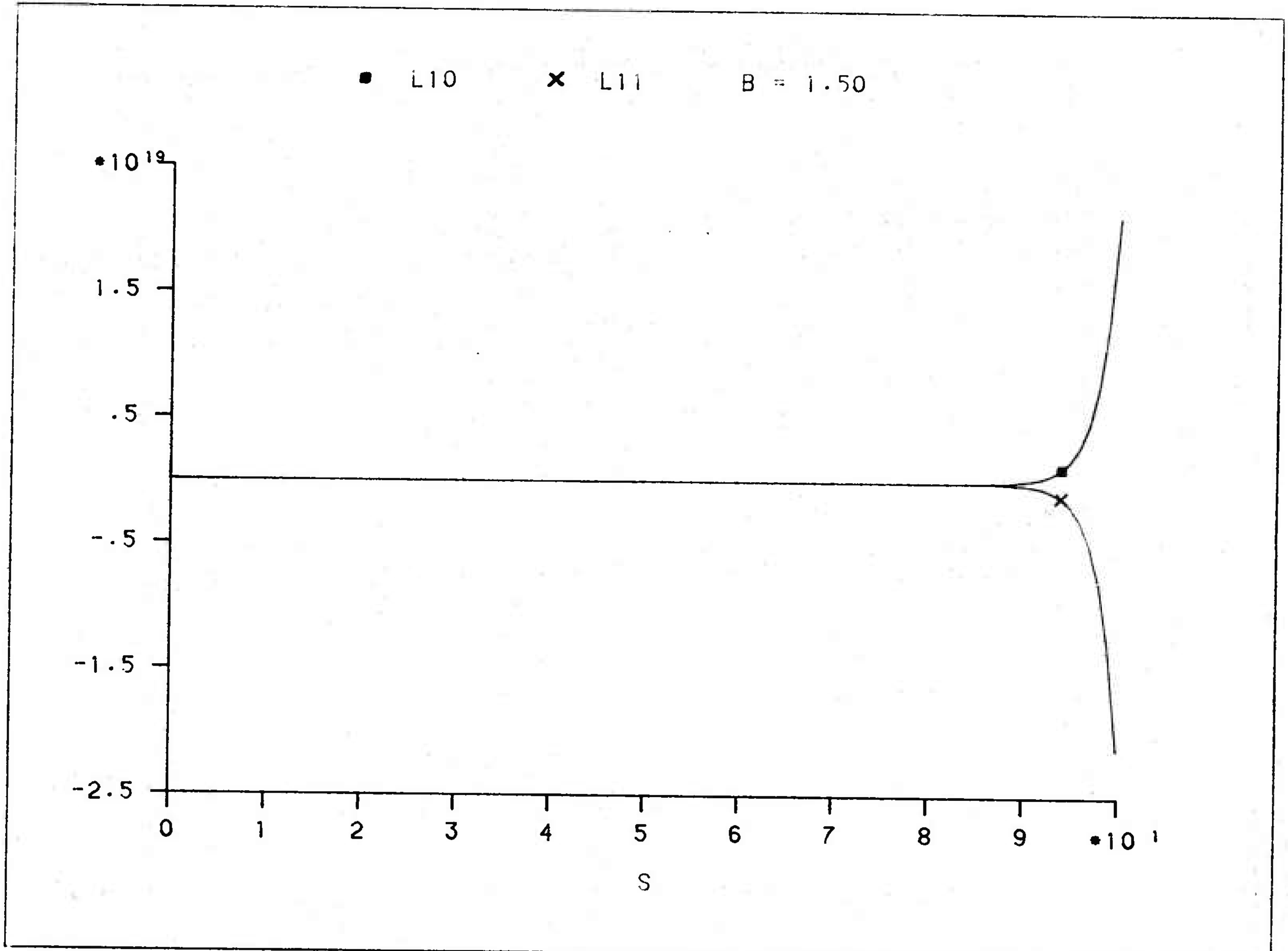

Figure 24. Graph of $L_{1,0}(s)$ and $L_{1,1}(s)$ for Wall Ratio = 1.5

expansion[7] of the determinant in Eq. (1); we note for instance

$$\begin{vmatrix} I_0(p) & \alpha_1 I_1(p) \\ I_1(p) & p\ I_0(p) \end{vmatrix} = p[I_0^2(p) - I_1^2(p)] - (2-2\nu)I_1^2(p)/p \quad . \tag{122}$$

Cancellation occurs in the bracketed term when $|p|$ is large.

Let

$$F_1 = \pi\ [z\ I_0^2(z) - z\ I_1^2(z)] \quad , \tag{123}$$

$$F_2 = [z\ I_0(z)\ K_0(z) + zI_1(z)\ K_1(z)] \quad , \tag{124}$$

$$F_3 \equiv [zK_0^2(z) - zK_1^2(z)]/\pi \quad , \tag{125}$$

$$F_4 = [zI_0(z)\ K_1(z) + zI_1(z)\ K_0(z)] \quad . \tag{126}$$

The last equation and the Wronskian relation

$$I_0(z)K_1(z) + I_1(z)K_0(z) = 1/z \tag{127}$$

give

$$F_4 = 1. \tag{128}$$

Derivatives of F_1, F_2, and F_3 are given by

$$F_1' = \pi\ [I_0^2(z) + I_1^2(z)] \quad , \tag{129}$$

$$F_1'' = \pi\ [4\ I_0(z)\ I_1(z) - 2I_1^2(z)/z] \quad , \tag{130}$$

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[7] *Thomas Muir, "A Treatise on the Theory of Determinants", Dover Publications NY, 1960. See p. 94 for expansion of a 4th order determinant.*

$$F_2' = [I_0(z)\ K_0(z) - I_1(z)\ K_1(z)] \quad , \tag{131}$$

$$F_2'' = 2\ [I_1(z)K_0(z)-K_1(z)I_0(z)+I_1(z)K_1(z)/z] \quad , \tag{132}$$

$$F_3' = [K_0^2(z) + K_1^2(z)]/\pi \quad , \tag{133}$$

$$F_3'' = -\ [4\ K_0(z)K_1(z)+2K_1^2(z)/z]/\pi \quad . \tag{134}$$

The squares and products of the individual Bessel functions are readily obtained from these formulas. We note that cancellation will occur in Eqs. (123), (125), (131), and (132) if $|z|$ is large.

Let

$$F=A_1F_1+A_2F_2+A_3F_3+A_4F_4 \quad ; \tag{135}$$

then F satisfies the third order differential equation

$$z^3F''' + 2z^2F'' - (4z^2+z)F' + F = A_4 \quad . \tag{136}$$

The functions F_1, F_2, and F_3 satisfy the homogeneous equation obtained by setting $A_4 = 0$; F_4 is a particular solution of the non-homogeneous equation.

Next, set sa = z in Eqs. (53) - (60), and define the functions

$$G_1(z) = P_0^2(z) - P_1^2(z) \quad , \tag{137}$$

$$G_2(z) = P_0(z)\ Q_0(z) + P_1(z)\ Q_1(z) \quad , \tag{138}$$

$$G_3(z) = Q_0^2(z) - Q_1^2(z) \quad , \tag{139}$$

$$G_4(z) = P_0(z)Q_1(z) + P_1(z)Q_0(z) \quad . \tag{140}$$

It is evident that

$$G_4(z) = 1 \tag{141}$$

on account of Eq. (127).

We also define the functions

$$H_1(z) = e^{2z}G_1(z) \quad , \tag{142}$$

$$H_2(z) = G_2(z) \quad , \tag{143}$$

$$H_3(z) = e^{-2z}G_3(z) \quad . \tag{144}$$

Then

$$F_1(z) = \frac{1}{2}\,[H_1(z) + 2i\,H_2(z) - H_3(z)] \quad , \tag{145}$$

$$F_2(z) = \frac{1}{2}\,[H_2(z) + iH_3(z)] \quad , \tag{146}$$

$$F_3(z) = \frac{1}{2}\,H_3(z) \quad . \tag{147}$$

It is clear that the functions H_1, H_2, and H_3 satisfy the differential equation

$$z^3H''' + 2z^2H'' - (4z^2+z)\,H' + H = 0 \quad , \tag{148}$$

since they are linear combinations of F_1, F_2, and F_3.

Assume

$$H = e^{\alpha z}\,G \quad , \tag{149}$$

$$z^3G''' + (3\alpha z^3+2z^2)G'' + [(3\alpha^2-4)\ z^3 + 4\ \alpha\ z^2-z]\ G'$$
$$+ [(\alpha^3-4\alpha)z^3 + (2\alpha^2z^2) - \alpha z + 1]\ G = 0 \quad . \tag{150}$$

We let α = 2, 0, and -2 in turn, obtaining the following differential equations:

$$z^3G_1''' + (6\ z^3+2z^2)G_1'' + (8z^3 + 8z^2-z)G_1' + (8z^2-2z+1)G_1 = 0 \quad , \tag{151}$$

$$z^3G_2''' + 2z^2G_2'' + (4z^3+z)G_2' + G_2 = 0 \quad , \tag{152}$$

$$z^3G_3''' + (-6z^3+12z^2)G_3''+(-8z^3-8z^2-z)G_3'+(8z^2+2z+1)G_3 = 0 \quad . \tag{153}$$

We assume

$$G_1(z) = a_{1,0} + a_{1,1}\ z^{-1} + a_{1,2}z^{-2} + \ldots \quad , \tag{154}$$

$$G_2(z) = a_{2,0} + a_{2,1}\ z^{-1} + a_{2,2}z^{-2} + \ldots \quad , \tag{155}$$

$$G_3(z) = a_{3,0} + a_{3,2}\ z^{-1} + a_{3,2}z^{-2} + \ldots \quad . \tag{156}$$

The coefficients are obtained by substituting each series in the appropriate equation, carrying out the required algebraic manipulation, and then setting the coefficients of each power of z equal to zero. The initial terms are obtained from the asymptotic expansions for $P_0(z)$, $Q_0(z)$, $P_1(z)$, and $Q_1(z)$. We find

$$(8k-8)a_{1,k}-(6k^2-14k+6)a_{1,k-1}+(k^3-5k^2+7k-3)a_{1,k-2} = 0 \quad , \tag{157}$$

$$a_{1,0} = 0, \quad a_{1,1} = \frac{1}{2}\ , \quad a_{1,2} = \frac{1}{8} \quad , \tag{158}$$

$$4ka_{2,k}-(k^3-5k^2+7k-3)a_{2,k-2} = 0 \quad , \tag{159}$$

$$a_{2,0} = 1, \quad a_{2,1} = 0, \quad a_{2,2} = -\frac{1}{8} \quad . \tag{160}$$

The coefficients of z^{-1}, z^{-3}, z^{-5} etc. in Eq. (155) are zero.

We note that

$$G_3(z) = G_1(-z) \quad , \tag{161}$$

so that

$$a_{3,k} = (-1)^k \, a_{1,k} \quad . \tag{162}$$

The derivatives of $G_1(z)$, $G_2(z)$, and $G_3(z)$ are obtained by differentiating the asymptotic series in Eq. (152), (153) and (154) term by term. Derivatives of $H_1(z)$, $H_2(z)$, and $H_3(z)$ are then obtained by differentiating Eqs. (141) and (143). The detailed derivatives and resulting formulas will not be given here.

Coefficients for these asymptotic series are listed in Table 10. The early coefficients, which are the most important, do not increase rapidly; consequently these series offer an effective method of computation. Characteristic roots for a thin-walled cylinder are listed in Tables 11 thru 15 for several wall ratios. These characteristic roots could not be obtained either with the original Bessel function subroutine or with the asymptotic series for the $L_{i,j}(s)$ functions.

When $|z|$ is large, it is unnecessary to calculate $F_1(z)$, $F_2(z)$, and $F_3(z)$, as the characteristic function $\Delta(s)$ can be expressed directly in terms of the P and Q functions with exponential factors. Combinations involving differences of nearly equal numbers can then be expressed in terms of the G functions. All similar determinants occurring in the stress analysis of thick-walled cylinders by Fourier methods can be treated in a similar manner. First, replace the modified Bessel functions in Eq. (1) with the appropriate Hankel asymptotic series. These are given in Eqs. (55) - (60) for the argument(sa); similar formulas hold for the argument (sb). Next, eliminate recessive terms like $ie^{-sa}Q_0(sa)/\sqrt{2\pi sa}$ by manipulating columns of the determinant. We finally obtain

TABLE 10. COEFFICIENTS FOR THE ASYMPTOTIC SERIES FOR THE SQUARES AND PRODUCTS OF MODIFIED BESSEL FUNCTIONS

N	A(1,N)	A(2,N)	A(3,N)
1	.5000000000000E+00	-.5000000000000E+00	0.
2	.1250000000000E+00	.1250000000000E+00	-.1250000000000E+00
3	.1406250000000E+00	-.1406250000000E+00	0.
4	.2226562500000E+00	.2226562500000E+00	-.7031250000000E-01
5	.4577636718750E+00	-.4577636718750E+00	0.
6	.1161804199219E+01	.1161804199219E+01	-.2197265625000E+00
7	.3515968322754E+01	-.3515968322754E+01	0.
8	.1237137794495E+02	.1237137794495E+02	-.1682281494141E+01
9	.4965300768614E+02	-.4965300768614E+02	0.
10	.2239406984299E+03	.2239406984299E+03	-.2384634017944E+02
11	.1121441469295E+04	-.1121441469295E+04	0.
12	.6174778305169E+04	.6174778305169E+04	-.5410138428211E+03
13	.3707904691611E+05	-.3707904691611E+05	0.
14	.2411637249036E+06	.2411637249036E+06	-.1795972738937E+05
15	.1688961699936E+07	-.1688961699936E+07	0.
16	.1267206354311E+08	.1267206354311E+08	-.8208156658421E+06
17	.1014078180404E+09	-.1014078180404E+09	0.
18	.8621844086271E+09	.8621844086271E+09	-.4941994321424E+08
19	.7761287181400E+10	-.7761287181400E+10	0.
20	.7374520284153E+11	.7374520284153E+11	-.3791127393823E+10
21	.7375620037140E+12	-.7375620037140E+12	0.
22	.7745388565802E+13	.7745388565802E+13	-.3609756412823E+12
23	.8520863821085E+14	-.8520863821085E+14	0.
24	.9799928340741E+15	.9799928340741E+15	-.4177164998963E+14
25	.1176089442421E+17	-.1176089442421E+17	0.

TABLE 10. COEFFICIENTS FOR THE ASYMPTOTIC SERIES FOR THE SQUARES AND PRODUCTS OF MODIFIED BESSEL FUNCTIONS (CONT'D)

N	A(1,N)	A(2,N)	A(3,N)
26	.1470219529546E+18	.1470219529546E+18	-.5773725659625E+16
27	.1911409156663E+19	-.1911409156663E+19	0.
28	.2580550761978E+20	.2580550761978E+20	-.9395191977380E+18
29	.3612956435360E+21	-.3612956435360E+21	0.
30	.5239027660123E+22	.5239027660123E+22	-.1777805201920E+21
31	.7858866430261E+23	-.7858866430261E+23	0.
32	.1218169766103E+25	.1218169766103E+25	-.3870754154086E+23
33	.1949137527733E+26	-.1949137527733E+26	0.
34	.3216175735203E+27	.3216175735203E+27	-.9608293344691E+25
35	.5467651858572E+28	-.5467651858572E+28	0.
36	.9568635648973E+29	.9568635648973E+29	-.2697328183744E+28
37	.1722394813152E+31	-.1722394813152E+31	0.
38	.3186499060212E+32	.3186499060212E+32	-.8502794731848E+30
39	.6054468333647E+33	-.6054468333647E+33	0.
40	.1180642941828E+35	.1180642941828E+35	-.2990698619526E+33
41	.2361325866705E+36	-.2361325866705E+36	0.
42	.4840793980628E+37	.4840793980628E+37	-.1167066730938E+36
43	.1016581544439E+39	-.1016581544439E+39	0.
44	.2185679932943E+40	.2185679932943E+40	-.5026941011685E+38
45	.4808556549135E+41	-.4808556549135E+41	0.
46	.1081937968149E+43	.1081937968149E+43	-.2378917872785E+41
47	.2488484724215E+44	-.2488484724215E+44	0.
48	.5847999369370E+45	.5847999369370E+45	-.1231647558043E+44
49	.1403533407366E+47	-.1403533407366E+47	0.
50	.3438688029955E+48	.3438688029955E+48	-.6949386599122E+46

TABLE 11. CHARACTERISTIC ROOTS FOR THIN-WALLED CYLINDERS, WALL RATIO = 1.010

a. REAL VALUES

ROOT NO	USING SINGLE PRECISION BESSEL FNS	USING DOUBLE PRECISION BESSEL FNS	USING DOUBLE PRECISION ASYMPTOTIC FNS
1	.266826954699 15E+01	.26682695469905737360167110D+01	.26682695469905737360167110D+01
25	.25188964108572E+02	.25188964110639463415075350D+02	.25188964110639463415075350D+02
50	.28701405550998E+02	.28701405510594562317782370D+02	.28701405510594562317782370D+02
75	.30744516824938E+02	.30744516869986900388086250D+02	.30744516869986900388086250D+02
100	.32190888487148E+02	.32190888449419361486621350D+02	.32190888449419361509974060D+02
125	.33311410971251E+02	.33311411043995284252884940D+02	.33311411043995284259205000D+02
150	.34226235555581E+02	.34226235849645788901292820D+02	.34226235849645788903468770D+02
175	.34999294775176E+02	.34999294741685838326091230D+02	.34999294741685857852561350D+02
200	.35668685220179E+02	.35668685403299426963694850D+02	.35668685403299438396714300D+02
225	.36258951960275E+02	.36258951944220434563988060D+02	.36258951944220441695797710D+02
250	.36786838254522E+02	.36786837582454477886758760D+02	.36786837582454482563190350D+02
275	.37264275675193E+02	.37264275998532247208723990D+02	.37264275998532250401525220D+02
300	.37700073640670E+02	.37700073475182097112848670D+02	.37700073475182099366712760D+02
325	.38100914679445E+02	.38100914405250586522874460D+02	.38100914405250588159224490D+02
350	.38471992264075E+02	.38471992589510454206745480D+02	.38471992589510455423560360D+02
375	.38817423622555E+02	.38817423926457000409283630D+02	.38817423926457001333083360D+02
400	.39140525039843E+02	.39140525479410630968924690D+02	.39140525479410631683065080D+02
425	.39444011509513E+02	.39444009695895476290317440D+02	.39444009695895476851280140D+02
450	.39730121149227E+02	.39730122969106968201787230D+02	.39730122969106968648741060D+02
475	.40000745260300E+02	.40000746648252478507101680D+02	.40000746648252478867813160D+02
500	.40257471550785E+02	.40257472084306432426050200D+02	.40257472084306432720571340D+02

TABLE 11. CHARACTERISTIC ROOTS FOR THIN-WALLED CYLINDERS, WALL RATIO = 1.010

b. IMAGINARY VALUES

ROOT NO	USING SINGLE PRECISION BESSEL FNS	USING DOUBLE PRECISION BESSEL FNS	USING DOUBLE PRECISION ASYMPTOTIC FNS
1	.28348046191925E+01	.2834804619194032621812180D+01	.2834804619194032621812180D+01
25	.38451927936893E+03	.3845192793706037325885255D+03	.3845192793706037325885255D+03
50	.77736028590678E+03	.7773602859083385176563480D+03	.7773602859083385176563480D+03
75	.11701123384017E+04	.1170112338373572818279525D+04	.1170112338373572818279525D+04
100	.15628396853164E+04	.1562839685295561767210404D+04	.1562839685295561766961313D+04
125	.19555565150439E+04	.1955556515032566566596481D+04	.1955556515032566566515144D+04
150	.23482678474305E+04	.2348267847416618592689133D+04	.2348267847416618592656527D+04
175	.27409759284687E+04	.2740975928399458901220704D+04	.2740975928399458902169526D+04
200	.31336819194761E+04	.3133681919446799352168219D+04	.3133681919446799352662023D+04
225	.35263864837826E+04	.3526386483917456411937888D+04	.3526386483917456412204335D+04
250	.39190900294088E+04	.3919090029261570954946681D+04	.3919090029261570955094043D+04
275	.43117928203336E+04	.4311792820079735153626090D+04	.4311792820079735153706325D+04
300	.47044950358248E+04	.4704495035952486164368636D+04	.4704495035952486164408519D+04
325	.50971968033380E+04	.5097196803203059864663131D+04	.5097196803203059864677309D+04
350	.54898982133305E+04	.5489898213369420569407894D+04	.5489898213369420569404785D+04
375	.58825993339730E+04	.5882599334463687260911122D+04	.5882599334463687260895762D+04
400	.62753002184071E+04	.6275300218110594174360105D+04	.6275300218110594174335612D+04
425	.66680009047674E+04	.6668000904227224223971061D+04	.6668000904227224223939435D+04
450	.70607014245949E+04	.7060701424180551854665649D+04	.7060701424180551854628213D+04
475	.74534018018687E+04	.7453401802972031545859936D+04	.7453401802972031545817581D+04
500	.78461020603579E+04	.7846102060782714773007537D+04	.7846102060782714772960858D+04

TABLE 12. CHARACTERISTIC ROOTS FOR THIN-WALLED CYLINDERS, WALL RATIO = 1.020

a. REAL VALUES

ROOT NO	USING SINGLE PRECISION BESSEL FNS	USING DOUBLE PRECISION BESSEL FNS	USING DOUBLE PRECISION ASYMPTOTIC FNS
1	.26682695469915E+01	.2668269546990573736016711D+01	.2668269546990573736016711D+01
25	.25188964108572E+02	.2518896411063946341507535D+02	.2518896411063946341507535D+02
50	.28701405550998E+02	.2870140551059456231778237D+02	.2870140551059456231778237D+02
75	.30744516824938E+02	.3074451686998690038808625D+02	.3074451686998690038808625D+02
100	.32190888487148E+02	.3219088844941936148662135D+02	.3219088844941936150997406D+02
125	.33311410971251E+02	.3331141104399528425288494D+02	.3331141104399528425920500D+02
150	.34226235555581E+02	.3422623584964578890129282D+02	.3422623584964578890346877D+02
175	.34999294775176E+02	.3499929474168583832609123D+02	.3499929474168585785256135D+02
200	.35668685220179E+02	.3566868540329942696369485D+02	.3566868540329943839671430D+02
225	.36258951960275E+02	.3625895194422043456398806D+02	.3625895194422044169579771D+02
250	.36786838254522E+02	.3678683758245447788675876D+02	.3678683758245448256319035D+02
275	.37264275675193E+02	.3726427599853224720872399D+02	.3726427599853225040152522D+02
300	.37700073640670E+02	.3770007347518209711284867D+02	.3770007347518209936671276D+02
325	.38100914679445E+02	.3810091440525058652287446D+02	.3810091440525058815922449D+02
350	.38471992264075E+02	.3847199258951045420674548D+02	.3847199258951045542356036D+02
375	.38817423622555E+02	.3881742392645700040928363D+02	.3881742392645700133308336D+02
400	.39140525039843E+02	.3914052547941063096892469D+02	.3914052547941063168306508D+02
425	.39444011509513E+02	.3944400969589547629031744D+02	.3944400969589547685128014D+02
450	.39730121149227E+02	.3973012296910696820178723D+02	.3973012296910696864874106D+02
475	.40000745260300E+02	.4000074666482524785071016 8D+02	.4000074666482524788678131 6D+02
500	.40257471550785E+02	.4025747208430643242605020D+02	.4025747208430643272057134D+02

TABLE 12. CHARACTERISTIC ROOTS FOR THIN-WALLED CYLINDERS, WALL RATIO - 1.020

b. IMAGINARY VALUES

ROOT NO	USING SINGLE PRECISION BESSEL FNS	USING DOUBLE PRECISION BESSEL FNS	USING DOUBLE PRECISION ASYMPTOTIC FNS
1	.28348046191925E+01	.2834804619194032621812180D+01	.2834804619194032621812180D+01
25	.38451927936893E+03	.3845192793706037325885255D+03	.3845192793706037325885255D+03
50	.77736028590678E+03	.7773602859083385176563480D+03	.7773602859083385176563480D+03
75	.11701123384017E+04	.1170112338373572818279525D+04	.1170112338373572818279525D+04
100	.15628396853164E+04	.1562839685295561767210404D+04	.1562839685295561766961313D+04
125	.19555565150439E+04	.1955556515032566566596481D+04	.1955556515032566566515144D+04
150	.23482678474305E+04	.2348267847416618592689133D+04	.2348267847416618592656527D+04
175	.27409759284687E+04	.2740975928399458901220704D+04	.2740975928399458902169526D+04
200	.31336819194761E+04	.3133681919446799352168219D+04	.3133681919446799352662023D+04
225	.35263864837826E+04	.3526386483917456411937888D+04	.3526386483917456412204335D+04
250	.39190900294088E+04	.3919090029261570954946681D+04	.3919090029261570955094043D+04
275	.43117928203336E+04	.4311792820079735153626090D+04	.4311792820079735153706325D+04
300	.47044950358248E+04	.4704495035952486164368636D+04	.4704495035952486164408519D+04
325	.50971968033380E+04	.5097196803203059864663131D+04	.5097196803203059864677309D+04
350	.54898982133305E+04	.5489898213369420569407894D+04	.5489898213369420569404785D+04
375	.58825993339730E+04	.5882599334463687260911122D+04	.5882599334463687260895762D+04
400	.62753002184071E+04	.6275300218110594174360105D+04	.6275300218110594174335612D+04
425	.66680009047674E+04	.6668000904227224223971061D+04	.6668000904227224223939435D+04
450	.70607014245949E+04	.7060701424180551854665649D+04	.7060701424180551854628213D+04
475	.74534018018687E+04	.7453401802972031545859936D+04	.7453401802972031545817581D+04
500	.78461020603579E+04	.7846102060782714773007537D+04	.7846102060782714772960858D+04

TABLE 13. CHARACTERISTIC ROOTS FOR THIN-WALLED CYLINDERS, WALL RATIO = 1.100

a. REAL VALUES

ROOT NO	USING SINGLE PRECISION BESSEL FNS	USING DOUBLE PRECISION BESSEL FNS	USING DOUBLE PRECISION ASYMPTOTIC FNS
1	.266826954699915E+01	.26682695469905737360167110D+01	.26682695469905737360167110D+01
25	.251889641085722E+02	.25188964110639463415075350D+02	.25188964110639463415075350D+02
50	.287014055509988E+02	.28701405510594562317782370D+02	.28701405510594562317782370D+02
75	.307445168249388E+02	.30744516869986900388086250D+02	.30744516869986900388086250D+02
100	.321908884871488E+02	.32190888449419361486621350D+02	.32190888449419361509974060D+02
125	.333114109712518E+02	.33311411043995284252884940D+02	.33311411043995284259205000D+02
150	.342262355555818E+02	.34226235849645788901292820D+02	.34226235849645788903468770D+02
175	.349992947751768E+02	.34999294741685838326091230D+02	.34999294741685857852561350D+02
200	.356686852201798E+02	.35668685403299426963694850D+02	.35668685403299438396714300D+02
225	.362589519602758E+02	.36258951944220434563988060D+02	.36258951944220441695797710D+02
250	.367868382545228E+02	.36786837582454477886758760D+02	.36786837582454482563190350D+02
275	.372642756751938E+02	.37264275998532247208723990D+02	.37264275998532250401525220D+02
300	.377000736406708E+02	.37700073475182097112848670D+02	.37700073475182099366712760D+02
325	.381009146794458E+02	.38100914405250586522874460D+02	.38100914405250588159224490D+02
350	.384719922640758E+02	.38471992589510454206745480D+02	.38471992589510455423560360D+02
375	.388174236225558E+02	.38817423926457000409283630D+02	.38817423926457001333083360D+02
400	.391405250398438E+02	.39140525479410630968924690D+02	.39140525479410631683065080D+02
425	.394440115095138E+02	.39444009695895476290317440D+02	.39444009695895476851280140D+02
450	.397301211492278E+02	.39730122969106968201787230D+02	.39730122969106968648741060D+02
475	.400007452603008E+02	.40000746648252478507101680D+02	.40000746648252478867813160D+02
500	.402574715507858E+02	.40257472084306432426050200D+02	.40257472084306432720571340D+02

TABLE 13. CHARACTERISTIC ROOTS FOR THIN-WALLED CYLINDERS, WALL RATIO - 1.100

b. IMAGINARY VALUES

ROOT NO	USING SINGLE PRECISION BESSEL FNS	USING DOUBLE PRECISION BESSEL FNS	USING DOUBLE PRECISION ASYMPTOTIC FNS
1	.283480461919 25E+01	.28348046191940326218121800+01	.28348046191940326218121800+01
25	.38451927936893E+03	.38451927937060373258852550+03	.38451927937060373258852550+03
50	.77736028590678E+03	.77736028590833851765634800+03	.77736028590833851765634800+03
75	.11701123384017E+04	.11701123383735728182795250+04	.11701123383735728182795250+04
100	.15628396853164E+04	.15628396852955617672104040+04	.15628396852955617669613130+04
125	.19555565150439E+04	.19555565150325665665964810+04	.19555565150325665665151440+04
150	.23482678474305E+04	.23482678474166185926891330+04	.23482678474166185926565270+04
175	.27409759284687E+04	.27409759283994589012207040+04	.27409759283994589021695260+04
200	.31336819194761E+04	.31336819194467993521682190+04	.31336819194467993526620230+04
225	.35263864837826E+04	.35263864839174564119378880+04	.35263864839174564122043350+04
250	.39190900294088E+04	.39190900292615709549466810+04	.39190900292615709550940430+04
275	.43117928203336E+04	.43117928200797351536260900+04	.43117928200797351537063250+04
300	.47044950358248E+04	.47044950359524861643686360+04	.47044950359524861644085190+04
325	.50971968033380E+04	.50971968032030598646631310+04	.50971968032030598646773090+04
350	.54898982133305E+04	.54898982133694205694078940+04	.54898982133694205694047850+04
375	.58825993339730E+04	.58825993344636872609111220+04	.58825993344636872608957620+04
400	.62753002184071E+04	.62753002181105941743601050+04	.62753002181105941743356120+04
425	.66680009047674E+04	.66680009042272242239710610+04	.66680009042272242239394350+04
450	.70607014245949E+04	.70607014241805518546656490+04	.70607014241805518546282130+04
475	.74534018018687E+04	.74534018029720315458599360+04	.74534018029720315458175810+04
500	.78461020603579E+04	.78461020607827147730075370+04	.78461020607827147729608580+04

TABLE 14. CHARACTERISTIC ROOTS FOR THIN-WALLED CYLINDERS, WALL RATIO = 1.200

a. REAL VALUES

ROOT NO	USING SINGLE PRECISION BESSEL FNS	USING DOUBLE PRECISION BESSEL FNS	USING DOUBLE PRECISION ASYMPTOTIC FNS
1	.266826954699 15E+01	.2668269546990573736016711D+01	.2668269546990573736016711D+01
25	.25188964108572E+02	.2518896411063946341507535D+02	.2518896411063946341507535D+02
50	.28701405550998E+02	.2870140551059456231778237D+02	.2870140551059456231778237D+02
75	.30744516824938E+02	.3074451686998690038808625D+02	.3074451686998690038808625D+02
100	.32190888487148E+02	.3219088844941936148662135D+02	.3219088844941936150997406D+02
125	.33311410971251E+02	.3331141104399528425288494D+02	.3331141104399528425920500D+02
150	.34226235555581E+02	.3422623584964578890129282D+02	.3422623584964578890346877D+02
175	.34999294775176E+02	.3499929474168583832609123D+02	.3499929474168585785256135D+02
200	.35668685220179E+02	.3566868540329942696369485D+02	.3566868540329943839671430D+02
225	.36258951960275E+02	.3625895194422043456398806D+02	.3625895194422044169579771D+02
250	.36786838254522E+02	.3678683758245447788675876D+02	.3678683758245448256319035D+02
275	.37264275675193E+02	.3726427599853224720872399D+02	.3726427599853225040152522D+02
300	.37700073640670E+02	.3770007347518209711284867D+02	.3770007347518209936671276D+02
325	.38100914679445E+02	.3810091440525058652287446D+02	.3810091440525058815922449D+02
350	.38471992264075E+02	.3847199258951045420674548D+02	.3847199258951045542356036D+02
375	.38817423622555E+02	.3881742392645700040928363D+02	.3881742392645700133308336D+02
400	.39140525039843E+02	.3914052547941063096892469D+02	.3914052547941063168306508D+02
425	.39444011509513E+02	.3944400969589547629031744D+02	.3944400969589547685128014D+02
450	.39730121149227E+02	.3973012296910696820178723D+02	.3973012296910696864874106D+02
475	.40000745260300E+02	.4000074666482524785071016 8D+02	.4000074666482524788678131 6D+02
500	.40257471550785E+02	.4025747208430643242605020D+02	.4025747208430643272057134D+02

TABLE 14. CHARACTERISTIC ROOTS FOR THIN-WALLED CYLINDERS, WALL RATIO = 1.200

b. IMAGINARY VALUES

ROOT NO	USING SINGLE PRECISION BESSEL FNS	USING DOUBLE PRECISION BESSEL FNS	USING DOUBLE PRECISION ASYMPTOTIC FNS
1	.283480461919 25E+01	.28348046191940326218121800+01	.28348046191940326218121800+01
25	.38451927936893E+03	.38451927937060373258852550+03	.38451927937060373258852550+03
50	.77736028590678E+03	.77736028590833851765634800+03	.77736028590833851765634800+03
75	.11701123384017E+04	.11701123383735728182795250+04	.11701123383735728182795250+04
100	.15628396853164E+04	.15628396852955617672104040+04	.15628396852955617669613130+04
125	.19555565150439E+04	.19555565150325665665964810+04	.19555565150325665665151440+04
150	.23482678474305E+04	.23482678474166185926891330+04	.23482678474166185926565270+04
175	.27409759284687E+04	.27409759283994589012207040+04	.27409759283994589021695260+04
200	.31336819194761E+04	.31336819194467993521682190+04	.31336819194467993526620230+04
225	.35263864837826E+04	.35263864839174564119378880+04	.35263864839174564122043350+04
250	.39190900294088E+04	.39190900292615709549466810+04	.39190900292615709550940430+04
275	.43117928203336E+04	.43117928200797351536260900+04	.43117928200797351537063250+04
300	.47044950358248E+04	.47044950359524861643686360+04	.47044950359524861644085190+04
325	.50971968033380E+04	.50971968032030598646631310+04	.50971968032030598646773090+04
350	.54898982133305E+04	.54898982133694205694078940+04	.54898982133694205694047850+04
375	.58825993339730E+04	.58825993344636872609111220+04	.58825993344636872608957620+04
400	.62753002184071E+04	.62753002181105941743601050+04	.62753002181105941743356120+04
425	.66680009047674E+04	.66680009042272242239710610+04	.66680009042272242239394350+04
450	.70607014245949E+04	.70607014241805518546656490+04	.70607014241805518546282130+04
475	.74534018018687E+04	.74534018029720315458599360+04	.74534018029720315458175810+04
500	.78461020603579E+04	.78461020607827147730075370+04	.78461020607827147729608580+04

TABLE 15. CHARACTERISTIC ROOTS FOR THIN-WALLED CYLINDERS, WALL RATIO = 1.250

a. REAL VALUES

ROOT NO	USING SINGLE PRECISION BESSEL FNS	USING DOUBLE PRECISION BESSEL FNS	USING DOUBLE PRECISION ASYMPTOTIC FNS
1	.23453003740550E+01	.2345300374055477190390249D+01	.2345300374055477190390249D+01
25	.20151158726000E+02	.2015115872627886368986157D+02	.2015115872627886368986157D+02
50	.22961120210380E+02	.2296112020609665912428179D+02	.2296112020609665912428179D+02
75	.24595611344311E+02	.2459561134903774169773925D+02	.2459561134903774169773925D+02
100	.25752709468891E+02	.2575270943981985920906906D+02	.2575270943981985920906906D+02
125	.26649127920222E+02	.2664912793476025580497035D+02	.2664912793476025580497035D+02
150	.27380987977240E+02	.2738098802266657773786657D+02	.2738098802266657773786657D+02
175	.27999435278140E+02	.2799943529086201053567884D+02	.2799943529086201053567884D+02
200	.28534947849769E+02	.2853494792480233760358250D+02	.2853494792480233760358250D+02
225	.29007161301145E+02	.2900716123188105322747177D+02	.2900716123188105322747177D+02
250	.29429469665557E+02	.2942946979729289065563386D+02	.2942946979729289065563386D+02
275	.29811420896951E+02	.2981142057181620523188190D+02	.2981142057181620523188190D+02
300	.30160058521723E+02	.3016005858558118732081303D+02	.3016005858558118732081303D+02
325	.30480731937958E+02	.3048073135542691226974368D+02	.3048073135542691226974368D+02
350	.30777594319169E+02	.3077759392369539551776055D+02	.3077759392369539551776055D+02
375	.31053939964971E+02	.3105393901037877776038497D+02	.3105393901037877928877671D+02
400	.31312420900947E+02	.3131242026698498849822607D+02	.3131242026698498967852834D+02
425	.31555208327844E+02	.3155520765215362288918084D+02	.3155520765215362381508411D+02
450	.31784097745521E+02	.3178409828090184587238537D+02	.3178409828090184660887334D+02
475	.32000595945838E+02	.3200059723294391592682578D+02	.3200059723294391651996213D+02
500	.32205977173247E+02	.3220597758932682637978939D+02	.3220597758932682686281333D+02

TABLE 15. CHARACTERISTIC ROOTS FOR THIN-WALLED CYLINDERS, WALL RATIO = 1.250

b. IMAGINARY VALUES

ROOT NO	USING SINGLE PRECISION BESSEL FNS	USING DOUBLE PRECISION BESSEL FNS	USING DOUBLE PRECISION ASYMPTOTIC FNS
1	.25256888034881E+01	.252568880348767114916405 2D+01	.2525688803487671149164052D+01
25	.30761593526337E+03	.3076159352632695131400719D+03	.3076159352632695131400719D+03
50	.62188848172954E+03	.6218884817245131733371932D+03	.6218884817245131733371932D+03
75	.93609003876297E+03	.9360900387683918248143566D+03	.9360900387683918248143566D+03
100	.12502718740654E+04	.1250271874071125673351126D+04	.1250271874071125673351126D+04
125	.15644453125953E+04	.1564445312590824269829213D+04	.1564445312590824269829213D+04
150	.18786143616649E+04	.1878614361680529082060034D+04	.1878614361680529082060034D+04
175	.21927808144822E+04	.2192780814468355217209429D+04	.2192780814468355217209429D+04
200	.25069455983738E+04	.2506945598315035650539494D+04	.2506945598315035650539494D+04
225	.28211092428219E+04	.2821109242902916784315229D+04	.2821109242902916784315229D+04
250	.31352720737100E+04	.3135272073590119484547502D+04	.3135272073590119484547502D+04
275	.34494343017671E+04	.3449434301674441220204616D+04	.3449434301674441220204616D+04
300	.37635960705233E+04	.3763596070565411820533832D+04	.3763596070565411820533832D+04
325	.40777574811857E+04	.4077757481145273854769246D+04	.4077757481145273854769246D+04
350	.43919186065513E+04	.4391918606518510406610090D+04	.4391918606518510406610090D+04
375	.47060795007400E+04	.4706079501002543899290306D+04	.4706079501002543899330775D+04
400	.50202402056668E+04	.5020240205827986316425214D+04	.5020240205827986316454750D+04
425	.53344007527169E+04	.5334400752875624484860362D+04	.5334400752875624484882335D+04
450	.56485611673686E+04	.5648561167197922619635440D+04	.5648561167197922619652062D+04
475	.59627214686627E+04	.5962721468763593997795704D+04	.5962721468763593997808473D+04
500	.62768816733103E+04	.6276881673691526081042134D+04	.6276881673691526081052065D+04

$$\Delta(s) = \begin{vmatrix} e^{sa}P_0(sa) & e^{-sa}Q_0(sa) & \alpha_1 e^{sa}P_1(sa) & \alpha_1 e^{-sa}Q_1(sa) \\ e^{sa}P_1(sa) & -e^{-sa}Q_1(sa) & sae^{sa}P_0(sa) & -sae^{-sa}Q_0(sa) \\ e^{sb}P_0(sb) & e^{-sb}Q_0(sb) & \beta_1 e^{sb}P_1(sb) & \beta_1 e^{-sb}Q_1(sb) \\ e^{sb}P_1(sb) & -e^{-sb}Q_1(sb) & sbe^{sb}P_0(sb) & -sbe^{-sb}Q_0(sb) \end{vmatrix} . \quad (163)$$

Reduction of this determinant by Laplace's method leads to the required result[8].

VI. SUMMARY AND CONCLUSIONS

In essence, these algorithms permit precise calculations in an extended region of the complex s plane, so the Fourier integrals can be evaluated either by quadratures or by residue theory. The ascending series for the $L_{i,j}(s)$ functions express $\Delta(s)$ and related determinants in a form which is free of logarithmic singularities. Consequently these formulas are valid in the entire finite part of the complex s plane; the cut along the negative real axis, needed for the logarithmic function, is not needed. The asymptotic expansions of the $L_{i,j}(s)$ functions eliminated exponential overrun and also the Stokes phenomena.[6] Finally the $F_i(z)$ functions eliminated cancellation of leading terms which occurred for large $|s|$.

It is clear that accurate values of the Bessel functions do not ensure accuracy in evaluating the Fourier integrals arising in thick-walled cylinder theory when $|s|$ is very large or very small; special attention for various combinations of Bessel functions is also required. The obvious alternative, the use of multiple precision calculations, is not standard practice at BRL and consequently was not considered.

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[8] *A.S. Elder, J.N. Walbert, K. Zimmerman, "Stresses near a Discontinuity of Loading in Thick and Thin Walled Cylinders", BRL Report in preparation.*

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